Longitudinal shear wave and transverse dilatational wave in solids

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Abstract: Dilatation wave involves compression and extension and is known as the curl-free solution of the elastodynamic equation. Shear wave on the contrary does not involve any change in volume and is the divergence-free solution. This letter seeks to examine the elastodynamic Green's function through this definition. By separating the Green's function in divergence-free and curl-free terms, it appears first that, strictly speaking, the longitudinal wave is not a pure dilatation wave and the transverse wave is neither a pure shear wave. Second, not only a longitudinal shear wave but also a transverse dilatational wave exists. These waves are shown to be a part of the solution known as coupling terms. Their special motion is carefully described and illustrated.

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1. Introduction

The anomalous polarization of shear wave was observed in seismology[1] in nondestructive testing[2,3] and medical imaging.[4] It is in this latter field called elastography that this special wave has been systematically studied[5,6] and even commercialized[7] for fibrosis diagnostic. In this paper, it is shown that this latter longitudinal shear wave has a symmetrical counterpart: The transverse dilatational wave. Let us consider the simplest situation of homogeneous linear isotropic elastic solid. In the literature, the question of the dilatational and shear waves is often treated in the frame of the homogeneous elastodynamic equation. Without any source term, the splitting of solutions into pure dilatational and shear waves does not raise special difficulties.[8] However this far-field approximation imposes a longitudinal polarization on the dilatational wave and a transverse polarization on the shear wave in contradiction with the experimental observations mentioned above. The exact full wave equation in the time domain with a source term is tackled in this letter:

$$(\lambda + 2\mu)\vec{\nabla}(\vec{\nabla}\vec{u}) - \mu\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{u} - \rho\frac{\partial^2\vec{u}}{\partial t^2} = \delta(r)\,\delta(t)\,\vec{n}. \quad (1)$$

λ and μ are the Lamé coefficients, $\vec{u}$ is the displacement vector, ρ is the density, and the point pulse force has the direction $\vec{n}$. The outgoing solution is the Green's function[9,10] and can be written in its harmonic form:[1]

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$$G_{mn}(0,r) = \frac{1}{4\pi\rho\alpha^2}\frac{\gamma_m\gamma_n}{r}e^{iqr} + \frac{1}{4\pi\rho\beta^2}\frac{\delta_{mn}-\gamma_m\gamma_n}{r}e^{ikr} + \frac{1}{4\pi\rho}\frac{3\gamma_m\gamma_n-\delta_{mn}}{r^3}\left[\frac{e^{ikr}}{i\omega}\left(\frac{r}{\beta}-\frac{1}{i\omega}\right)-\frac{e^{iqr}}{i\omega}\left(\frac{r}{\alpha}-\frac{1}{i\omega}\right)\right]. \quad (2)$$

In this expression, α and β are the longitudinal and transverse wave speeds, r is the radial distance to the source, q and k are the dilatational and shear wave numbers, respectively, ω is the angular frequency, and the direction cosines $\gamma_i = x_i/r$ (see Fig. 1). Each term of the sum can be identified as a longitudinal far field wave G^{L}, a transverse far field wave G^{T}, and a near field wave G^{NF} that has sometimes been identified as a coupling wave,

$$G_{mn}(0,r) = G^{\mathrm{L}}_{mn} + G^{\mathrm{T}}_{mn} + G^{\mathrm{NF}}_{mn}. \quad (3)$$

2. The two terms Green's function

It is worth noticing that the longitudinal far field wave is not strictly curl-free. Without loss of generality, let us consider now a source oriented in the direction 3:

$$\nabla\times G^{\mathrm{L}}_{3n} = \left(\frac{\partial G^{\mathrm{L}}_{33}}{\partial x_2}-\frac{\partial G^{\mathrm{L}}_{32}}{\partial x_3}\right)\hat{\mathbf{1}} - \left(\frac{\partial G^{\mathrm{L}}_{33}}{\partial x_1}-\frac{\partial G^{\mathrm{L}}_{31}}{\partial x_3}\right)\hat{\mathbf{2}} + \left(\frac{\partial G^{\mathrm{L}}_{32}}{\partial x_1}-\frac{\partial G^{\mathrm{L}}_{31}}{\partial x_2}\right)\hat{\mathbf{3}} = \frac{e^{iqr}}{4\pi\rho\alpha^2 r^2}\left[-\gamma_2\hat{\mathbf{1}}+\gamma_1\hat{\mathbf{2}}\right].$$

Equivalently, the transverse far field wave is not divergence-free:

$$\nabla\cdot G^{\mathrm{T}}_{3n} = \frac{\partial G^{\mathrm{T}}_{31}}{\partial x_1}+\frac{\partial G^{\mathrm{T}}_{32}}{\partial x_2}+\frac{\partial G^{\mathrm{T}}_{33}}{\partial x_3} = \frac{-e^{ikr}}{2\pi\rho\beta^2 r^2}\gamma_3.$$

What remains tends to zero in the far field as r^2. In a more remarkable way, it is perfectly compensated by the divergence and curl of the coupling term, respectively:[11]

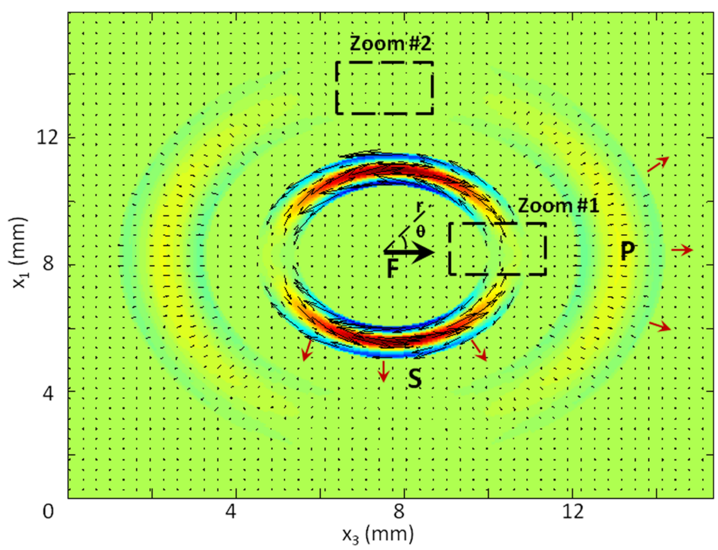

Fig. 1. (Color online) Elastic Green's function. The dilatational and the shear wave are clearly visible at $\theta = 0$ and $\theta = \pi/2$. The longitudinal shear wave and the transverse dilatational wave are located inside the dashed boxes called Zoom #1 and #2.

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$$\nabla \wedge G_{3n}^{\rm NF} = \frac{e^{ikr}}{4\pi\rho\beta^2 r^2}\left[-\gamma_2 \hat{\mathbf{1}} + \gamma_1 \hat{\mathbf{2}}\right] - \frac{e^{iqr}}{4\pi\rho\alpha^2 r^2}\left[-\gamma_2 \hat{\mathbf{1}} + \gamma_1 \hat{\mathbf{2}}\right], \tag{4}$$

$$\nabla \cdot G_{3n}^{\rm NF} = \frac{e^{ikr}}{2\pi\rho\beta^2 r^2}\gamma_3 - \frac{e^{iqr}}{2\pi\rho\alpha^2 r^2}\gamma_3. \tag{5}$$

The fact that the near field term is neither curl-free nor divergence-free is probably a strong argument in favor of the name "coupling" between shear and dilatational waves. However, the near field term can be split in $G^{\rm NFS}$ $G^{\rm NFP}$:

$$\begin{cases} G_{mn}^{\rm NFP}(0,r) = \dfrac{1}{4\pi\rho}\dfrac{3\gamma_m\gamma_n - \delta_{mn}}{r^3}\dfrac{e^{iqr}}{i\omega}\left(\dfrac{r}{\alpha} - \dfrac{1}{i\omega}\right), \\ G_{mn}^{\rm NFS}(0,r) = \dfrac{1}{4\pi\rho}\dfrac{3\gamma_m\gamma_n - \delta_{mn}}{r^3}\dfrac{e^{ikr}}{i\omega}\left(\dfrac{r}{\beta} - \dfrac{1}{i\omega}\right), \end{cases} \tag{6}$$

such that

$$\begin{cases} \nabla(G_{mn}^{\rm T} + G_{mn}^{\rm NFS}) = 0, \\ \nabla \wedge (G_{mn}^{\rm L} + G_{mn}^{\rm NFP}) = 0. \end{cases}$$

The Green's function can thus be rewritten as

$$\begin{aligned} G_{mn}(0,r) &= \frac{1}{4\pi\rho\alpha^2}\frac{\gamma_m\gamma_n}{r}e^{iqr} - \frac{1}{4\pi\rho}\frac{3\gamma_m\gamma_n - \delta_{mn}}{r^3}\frac{e^{iqr}}{i\omega}\left(\frac{r}{\alpha} - \frac{1}{i\omega}\right) \\ &\quad + \frac{1}{4\pi\rho\beta^2}\frac{\delta_{mn} - \gamma_m\gamma_n}{r}e^{ikr} + \frac{1}{4\pi\rho}\frac{3\gamma_m\gamma_n - \delta_{mn}}{r^3}\frac{e^{ikr}}{i\omega}\left(\frac{r}{\beta} - \frac{1}{i\omega}\right), \\ G_{mn}(0,r) &= \underbrace{G_{mn}^{\rm L} + G_{mn}^{\rm NFP}}_{P} + \underbrace{G_{mn}^{\rm T} + G_{mn}^{\rm NFS}}_{S} = G_{mn}^{\rm P} + G_{mn}^{\rm S}. \end{aligned} \tag{7}$$

This form of the Green's function (7) does not change anything to the result but highlights the two distinct parts of the Green's function: The curl-free term $G_{mn}^{\rm P}$ and divergence-free term $G_{mn}^{\rm S}$. They can rigorously be interpreted as dilatational P and shear S waves. An important comment can be made at this point. The term "Coupling" should probably be abandoned since it can be split to be part of two independent propagating waves. "Near field" is more appropriate for the $G^{\rm NFS}$ and $G^{\rm NFP}$ terms that obey to a $1/r^2$ decreasing law. They nonetheless can be detected in the far field. The reason is that specific directions exist to which $G^{\rm L}$ or $G^{\rm T}$ are zero. They can no more mask the influence of $G^{\rm NFP}$ and $G^{\rm NFS}$. Along these directions, whatever the distance from the source and thus in the far field, these latter terms give rise to detectable waves. This statement is only true for pulsed sources able to create well-separated S and P wave-fronts. This is the reason why, while the theoretical analyses are conducted on harmonic Green's function, the illustrations use pulsed Green's function. For example, in elastography the term $G^{\rm CS}$ is frequently measured in the direction where $G^{\rm T}$ is zero that is to say along a pulsed force. Without any doubt this special longitudinally polarized wave is a shear wave. Now let us take a closer look to these special angles $\theta = \pi/2$ and $\theta = 0$ according to which the near field terms are detectable.

3. Divergence, curl, and polarization of near field waves

In the direction perpendicular to the point force, the angle θ is $\pi/2$, Fig. 1. From Eqs. (2) and (3) the longitudinal far field wave gives $G_{3n}^{\rm L}(\theta = \pi/2) = 0$. According to Eq. (6), the components of the dilatational wave in the same direction reduce to $G_{33}^{\rm NFP}(\theta = \pi/2) = -(1/4\pi\rho r^3)(e^{iqr}/i\omega)(r/\alpha - 1/i\omega)$ and $G_{31}^{\rm NFP}(\theta = \pi/2) = G_{32}^{CP}(\theta = \pi/2) = 0$. As a consequence, in this direction the dilatational wave has a transverse

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polarization. Furthermore, in this direction the cosine director $\gamma_3(\theta = \pi/2) = 0$ thus, following Eq. (5) $\nabla \cdot G_{3n}^{\mathrm{NF}}(\mathbf{0}, \mathbf{r}) = 0$. This dilatational wave is divergence-free in this direction.

The same logic holds for the shear wave. In the direction parallel to the point force, the angle θ is 0. From Eqs. (2) and (3), the transverse far field wave is $G_{3n}^{\mathrm{T}}(\theta = 0) = 0$. According to Eq. (6), the components of the shear wave in the same direction reduce to $G_{33}^{\mathrm{NFS}}(\theta = 0) = -(1/2\pi\rho\, r^3)(\mathrm{e}^{ikr}/i\omega)(r/\beta - 1/i\omega)$ and $G_{31}^{\mathrm{NFS}}(\theta = 0) = G_{32}^{\mathrm{NFS}}(\theta = 0) = 0$. As a consequence the shear wave has a longitudinal polarization. In addition, the cosine directors $\gamma_1(\theta = 0) = \gamma_2(\theta = 0) = 0$ thus, according to Eq. (4) $\nabla \wedge G_{3n}^{\mathrm{NF}}(\mathbf{0}, \mathbf{r}) = \vec{0}$. This shear wave is curl-free in this direction.

4. Numerical Green's function

These two special waves are locally and simultaneously divergence- and curl-free. What kind of strain can one expect from such properties? In order to illustrate this question, the Green's function in an infinite homogeneous elastic medium was numerically computed and represented in the following figures. The chosen parameters without loss of generality are 2000 and 1000 m s^{-1} for the dilatational wave α and the shear wave speed β, 1 MHz for the central frequency pulse, 15 mm for the spatial dimension and the axis 3, which is the direction of the point source, is set horizontally. In Fig. 1, a typical radiation pattern is represented with a dilatational wave clearly ahead and behind the source and with a shear wave on both sides. The two regions inside the dashed boxes have been zoomed in Figs. 2 and 3. They are the center of interest of the paper.

In the region Zoom #1 represented in Fig. 2, the depth range corresponds to the S-wave front. A rectangle is super imposed to the actual displacement field in order to amplify and to clearly illustrate the strain induced by the longitudinal shear wave. This first rectangle on the left is at rest, then the second rectangle is stretched vertically, the third is at rest again, the fourth is stretched horizontally, and the last one comes back at rest. The whole strain sequence is represented on the upper right panel

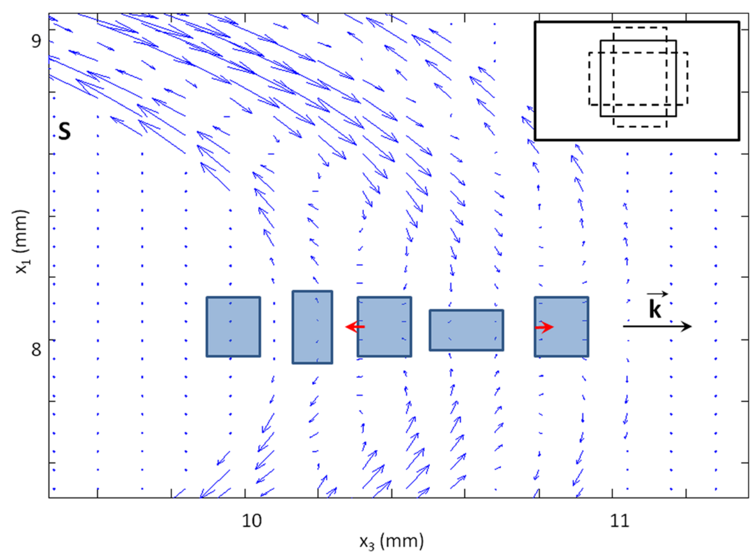

Fig. 2. (Color online) Zoom #1. The longitudinal shear wave is submitted to a special strain field propagating at the shear speed with a wave vector $\vec{k}$. It is represented as a semi-transparent rectangle. The longitudinal component of displacements along the horizontal axis of symmetry is clearly apparent (black and red arrows). The strain sequence illustrated in the upper right panel implies neither volumic change nor particle velocity circulation: It is divergence- and curl-free.

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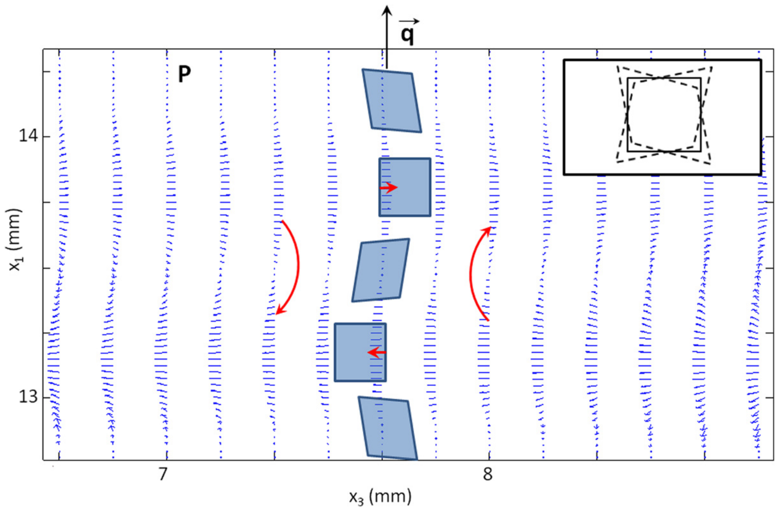

Fig. 3. (Color online) Zoom #2. The transverse dilatation wave propagates at the dilatation wave speed with a wave vector $\vec{q}$. The transversal component of displacements along the vertical axis of symmetry is clearly apparent (black and red arrows). The strain sequence illustrated in the upper right panel implies neither volumic change nor particle velocity circulation: It is divergence- and curl-free.

of Fig. 2. Along the axis of symmetry a longitudinal component is clearly apparent. It is quite straightforward to verify that this strain sequence induces a longitudinal polarization of the S-wave on one hand and implies neither volumic change nor particle velocity circulation on the other hand. As a consequence, it is divergence- and curl-free. This does not impose the strain tensor to be zero: The induced shape changes of a rhombus inside the five rectangles prove that a shear strain is still present. At last, we may point out that in 1990, Yamakoshi *et al.*[4] came to the same description of the longitudinal S-wave without Green's function, but with a remarkable intuition.

As far as the transverse dilatational wave is concerned, the strain is more complex. In the region of Zoom #2 represented in Fig. 3, the depth range corresponds to the dilatational wave front. As seen on the upper right panel, the shape of the rectangles is impacted with respect to the symmetry of the diagonals. In contrast with simple shear, this strain sequence known as pure shear leaves unchanged the rotation and the volume of rectangles.[12] More obvious is the global displacement field in the direction which is perpendicular to the propagation: The dilatational wave has a transverse polarization and is divergence- and curl-free.

Some practical applications of this work could consist of using a longitudinal transducer or a laser to measure both P- and S-wave arrivals from a point source. Equivalently a transverse transducer is able to detect both waves if placed in the correct direction. Thus one measurement only can supply elastic properties of solids.

Acknowledgment

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