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A REPORT ON REALIZABILITY

WALTER FERRER SANTOS, MAURICIO GUILLERMO, AND OCTAVIO MALHERBE

Abstract. Besides recalling the basic definitions of Realizability Lattices, Abstract Krivine Structures, Ordered Combinatory Algebras and Tripos and reviewing its relationships, we propose a new foundational framework for realizability. Motivated by Streicher's paper *Krivine's Classical Realizability from a Categorical Perspective* [9], we define the concept of *Krivine's Ordered Combinatory Algebras* (${}^{\mathcal{K}}\mathcal{OCA}$) as a common platform that is strong enough to do both: categorical and computational semantics. The $\mathcal{OCA}$s produced by Streicher from $\mathcal{AKS}$s in [9] are particular cases of ${}^{\mathcal{K}}\mathcal{OCA}$s.

1. Introduction

In this report we revisit the important construction presented in the paper: *Krivine's Classical Realizability from a Categorical Perspective* by Thomas Streicher –see [9]–.

As the results of Streicher's paper are the basis of our presentation as well as of our contributions, we cite its Introduction in some length.

Thereat, the author states: *In a sequence of papers* ([5];[6]; [8]) *J.-L. Krivine has introduced his notion of Classical Realizability for classical second order logic and Zermelo-Fraenkel set theory. Moreover, in more recent work* ([7]) *he has considered forcing constructions on top of it with the ultimate aim of providing a realizability interpretation for the axiom of choice. The aim of this paper is to show how Krivine's classical realizability can be understood as an instance of the categorical approach to realizability as started by Martin Hyland in* ([4]) *and described in detail in* ([10]).

Later he mentions that the main purpose of his construction, is to: (c.f. [9]) *Introduce a notion of abstract Krivine structure (aks) and show how to construct a classical realizability model for each such aks* $[\cdots$ and$]$ *show how any aks A gives rise to an order combinatory algebra (oca) with a filter of distinguished truth values which induces a tripos (see* ([10]; [2]) *for explanation of these notions) which also gives rise to a model of ZF*.

In this report, in Sections 2, 3 and 4, we start with a recapitulation of the main constructions of Streicher introducing the concept of $\mathcal{AKS}$ –the Abstract Krivine structures mentioned before– in a modular step by step manner, that we hope makes the subject easier to digest.

In Sections 5, 6 and 7, besides recalling the definition of combinatory algebra and ordered combinatory algebra, we introduce the notion of adjunctor, that is an element e of the algebra that (if $\circ$ is the application and $\rightarrow$ the implication of the algebra) guarantees that: for all $a, b, c \in A$, if $a \circ b \leq c$, then $e \circ a \leq (b \rightarrow c)$. We also show that an Abstract Krivine Structure in the sense of [9], produces an ordered combinatory algebra with application, implication and adjunctor.

In Section 8, we show that –with the addition of a completeness condition with respect to the inf of arbitrary subsets to the ordered combinatory algebras considered above– we can induce a tripos *directly* from the algebra, with no need to first walk back to the –a priori richer– abstract Krivine structure.

In Section 9, we show that we can define Realizability for high order languages in the class of $\mathcal{OCA}$s considered in the above section. In particular this means that we can define Realizability for high order arithmetics. In conclusion in this set up we can do both semantics: computational and categorical.

We would like to thank Jonas Frey and Alexandre Miquel, for sharing with us their deep expertise on the subject, when visiting Uruguay in 2013.

In a joint paper that is currently in preparation, more thorough results of this collaboration will be presented.

2. A basic set theoretical construction: Realizability Lattices.

1. We consider the following set theoretical data.

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Definition 2.1. A realizability lattice –abbreviated as $\mathcal{RL}$– is a triple $(\Lambda, \Pi, \perp\!\!\!\perp)$ where Λ and Π are sets and $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$ is a subset. The elements of Λ are called *terms* and the elements of Π are called *stacks*.

(1) If $t \star \pi \in \perp\!\!\!\perp$, we write that $t \perp \pi$ and say that t is perpendicular to π or that t realizes $\{\pi\}$.

(2) Given $P \subseteq \Pi$ and $L \subseteq \Lambda$, we define

$$^{\perp}P = \{t \in \Lambda : t \perp \pi\,,\ \forall \pi \in P\} \subseteq \Lambda \quad , \quad L^{\perp} = \{\pi \in \Pi : t \perp \pi\,,\ \forall t \in L\} \subseteq \Pi.$$

(3) If $t \in {}^{\perp}P$, we say that t realizes P and write $t \models P$. In other words t realizes P if and only if $t \perp \pi$ for all $\pi \in P$.

2. The following definitions can be established for an $\mathcal{RL}$.

Definition 2.2. Given $(\Lambda, \Pi, \perp\!\!\!\perp)$ an $\mathcal{RL}$, we define a pair of maps:

$$\begin{aligned}
(\quad)^{\perp} : \mathcal{P}(\Lambda) &\longrightarrow \mathcal{P}(\Pi)\\
\Lambda \supseteq L &\longrightarrow L^{\perp} = \{\pi \in \Pi |\ \forall t \in L, t \star \pi \in \perp\!\!\!\perp\} = \{\pi \in \Pi |\ L \times \{\pi\} \subseteq \perp\!\!\!\perp\} \subseteq \Pi;
\end{aligned}$$

$$\begin{aligned}
{}^{\perp}(\quad) : \mathcal{P}(\Pi) &\longrightarrow \mathcal{P}(\Lambda)\\
\Pi \supseteq P &\longrightarrow {}^{\perp}P = \{t \in \Lambda |\ \forall \pi \in P, t \star \pi \in \perp\!\!\!\perp\} = \{t \in \Lambda |\ \{t\} \times P \subseteq \perp\!\!\!\perp\} \subseteq \Lambda.
\end{aligned}$$

The pairs of $\Lambda \times \Pi$ are called *processes* and it is customary to denote the process (t, π) as $t \star \pi$.

Observation 2.3. In the notations above for an $\mathcal{RL}$ one has that:

(1) The maps $L \to L^{\perp}$ and $P \to {}^{\perp}P$ are antimonotone with respect to the order given by the inclusion of sets and ${}^{\perp}\emptyset = \Lambda$ and $\emptyset^{\perp} = \Pi$.

(2) Let us consider the behaviour of the operators $(\quad)^{\perp}$ and ${}^{\perp}(\quad)$ with respect to the lattice structure of the domain and codomain. We have that for $P_i \subseteq \Pi, i \in I$ and $L_i \subseteq \Lambda, i \in I$:

$$^{\perp}\Big(\bigcap_{i\in I} P_i\Big) \supseteq \bigcup_{i\in I} {}^{\perp}P_i\,,\ {}^{\perp}\Big(\bigcup_{i\in I} P_i\Big) = \bigcap_{i\in I} {}^{\perp}P_i;$$
$$\Big(\bigcap_{i\in I} L_i\Big)^{\perp} \supseteq \bigcup_{i\in I} L_i^{\perp}\,,\ \Big(\bigcup_{i\in I} L_i\Big)^{\perp} = \bigcap_{i\in I} L_i^{\perp}.$$

(3) For an arbitrary $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$, one has that ${}^{\perp}(L^{\perp}) \supseteq L$ and $({}^{\perp}P)^{\perp} \supseteq P$.

(4) One has that $({}^{\perp}\Pi)^{\perp} = \Pi$ and ${}^{\perp}(\Lambda^{\perp}) = \Lambda$. Notice that in general it may happen that ${}^{\perp}\Pi \neq \emptyset$ or $\Lambda^{\perp} \neq \emptyset$ –see later Observation 4.2,(4).

(5) For an arbitrary $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$, one has that $({}^{\perp}(L^{\perp}))^{\perp} = L^{\perp}$ and ${}^{\perp}(({}^{\perp}P)^{\perp}) = {}^{\perp}P$.

Proof. The proof of the first four properties is immediate. For the fifth one, applying the $\perp$ operator in ${}^{\perp}(L^{\perp}) \supseteq L$ we obtain that $({}^{\perp}(L^{\perp}))^{\perp} \subseteq L^{\perp}$and substituting in the inequality $({}^{\perp}P)^{\perp} \supseteq P$, the subset P by $L^{\perp}$ we obtain the reverse inclusion. Similarly for subsets $P \subseteq \Pi$. □

3. In the above context, the following definition is natural.

Definition 2.4. In the situation that we have an $\mathcal{RL}$ as above, we define the following sets:

$$\mathcal{P}_{\perp}(\Lambda) = \{L \subseteq \Lambda |\, {}^{\perp}(L^{\perp}) = L\} \subseteq \mathcal{P}(\Lambda),$$
$$\mathcal{P}_{\perp}(\Pi) = \{P \subseteq \Pi |\, ({}^{\perp}P)^{\perp} = P\} \subseteq \mathcal{P}(\Pi)$$

Notice that the only relevant structure at this point is the *lattice* structure in the sets $P_{\perp}(\Lambda)$ and $P_{\perp}(\Pi)$, where we take the (set theoretical) inclusion as the order and as “meet” and “join” the intersection and union respectively follwed by taking double perpendicularity.

Lemma 2.5. *In the above context of an $\mathcal{RL}$ the maps $(\quad)^{\perp} : \mathcal{P}(\Lambda) \to \mathcal{P}(\Pi)$ and ${}^{\perp}(\quad) : \mathcal{P}(\Pi) \to \mathcal{P}(\Lambda)$ when restricted respectively to $\mathcal{P}_{\perp}(\Lambda)$ and $\mathcal{P}_{\perp}(\Pi)$ are order reversing isomorphisms inverse of each other. Moreover with respect to the order given by the inclusion, $\Lambda^{\perp}$ and Π; ${}^{\perp}\Pi$ and Λ are the minimal and maximal elements of $\mathcal{P}_{\perp}(\Pi)$ and $\mathcal{P}_{\perp}(\Lambda)$ respectively.*

Proof. This result follows immediately from the previous considerations and it is in fact a general result concerning a Galois connection.

Indeed, it is clear that $\mathrm{Im}((\ \ )^{\perp}) = \mathcal{P}_{\perp}(\Pi)$ and $\mathrm{Im}(^{\perp}(\ \ )) = \mathcal{P}_{\perp}(\Lambda)$.

By the very definition of $\mathcal{P}_{\perp}(\Lambda)$ it is clear that if we apply succesively the maps $(\ \ )^{\perp} : \mathcal{P}(\Lambda) \to \mathcal{P}(\Pi)$ and $^{\perp}(\ \ ) : \mathcal{P}(\Pi) \to \mathcal{P}(\Lambda)$ to $L \in \mathcal{P}_{\perp}(\Lambda)$ we obtain again L. Similarly for $P \in \mathcal{P}_{\perp}(\Pi)$. □

The following observation will be used repeatedly.

Observation 2.6. The following results are valid in an $\mathcal{RL}$. Notice that the last three assertions need stronger hypothesis than the first.

(1) If $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$, then $L \subseteq {}^{\perp}P$ if and only if $P \subseteq L^{\perp}$.
(2) If $L \in \mathcal{P}_{\perp}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$, then $L^{\perp} \subseteq P$ implies that ${}^{\perp}P \subseteq L$.
(3) If $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}_{\perp}(\Pi)$, then ${}^{\perp}P \subseteq L$ implies that $L^{\perp} \subseteq P$.
(4) If $L \in \mathcal{P}_{\perp}(\Lambda)$ and $P \in \mathcal{P}_{\perp}(\Pi)$, then $L^{\perp} \subseteq P$ if and only if ${}^{\perp}P \subseteq L$.

As in the general situation of a Galois connection, the above conditions (1) and (4) can be read as adjunction relations between the functors ${}^{\perp}(-)$ and $(-)^{\perp}$ in the adequate domain and codomain.

3. The push map in a realizabilty lattice

4. In this section we add what we call *a push map* to a realizability lattice, with which we can add the first elements of a *calculus* to our structure.

Definition 3.1. A map $(t,\pi) \mapsto t.\pi : \Lambda \times \Pi \to \Pi$ defined in a realizability lattice $(\Lambda, \Pi, \perp\!\!\!\perp)$, will be called a *push* map and denoted as $\mathrm{push}(t,\pi) = t.\pi$. In that case we say that the realizability lattice is endowed with a push map.

Definition 3.2. For an $\mathcal{RL}$ with a push, for $L \subseteq \Lambda$ and $P \subseteq \Pi$ we define:

$$L \rightsquigarrow P = \{\pi \in \Pi : L.\pi \subseteq P\} \subseteq \Pi \quad \text{right conductor of } L \text{ into } P.$$

Notice that:

$$L \rightsquigarrow P = \bigcup \{Q \subseteq \Pi : L.Q \subseteq P\}.$$

We can use the push map in order to define a map:

$$(L, P) \mapsto L.P : \mathcal{P}(\Lambda) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi),$$

that combined with the operators $(\ \ )^{\perp}$ and $^{\perp}(\ \ )$ yields natural binary operations in $\mathcal{P}_{\perp}(\Lambda)$ and $\mathcal{P}_{\perp}(\Pi)$.

Observation 3.3. We can interpret the maps in Definition 3.2 as follows. Consider $L \subseteq \Lambda$ and define $a_L, m_L : \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ as $a_L(P) = L \rightsquigarrow P$ and $m_L(P) = L.P$[1]. In this notation the following "adjunction relations" holds: For all $P, Q \subseteq \Pi$:

$$m_L(Q) \subseteq P \Leftrightarrow Q \subseteq a_L(P).$$

Definition 3.4. We define the following binary operations in $\mathcal{P}_{\perp}(\Pi)$. Let $P, Q \in \mathcal{P}_{\perp}(\Pi)$:

(1) $$P \circ Q = ({}^{\perp}\{\pi \in \Pi : {}^{\perp}Q.\pi \subseteq P\})^{\perp} = ({}^{\perp}({}^{\perp}Q \rightsquigarrow P))^{\perp} \in \mathcal{P}_{\perp}(\Pi).$$

(2) $$P \to Q = ({}^{\perp}\,\mathrm{push}({}^{\perp}P, Q))^{\perp} = ({}^{\perp}({}^{\perp}P \cdot Q))^{\perp} \in \mathcal{P}_{\perp}(\Pi).$$

Observation 3.5. (1) Observe that in accordance to the above Definition 3.4, (1), we have that for $P, Q \in \mathcal{P}_{\perp}(\Pi)$:

$$P \subseteq ({}^{\perp}Q \cdot P) \circ Q.^{2}$$

(2) Notice that: $P \circ Q = ({}^{\perp}\{\pi \in \Pi : {}^{\perp}P \subseteq {}^{\perp}({}^{\perp}Q.\pi)\})^{\perp}$.
(3) From the definition of $P \to Q$, we deduce that ${}^{\perp}(P \to Q) = {}^{\perp}({}^{\perp}P.Q)$.

[1] In principle, the maps defined above are not internal maps in the corresponding $\mathcal{P}_{\perp}$s.

[2] Notice the slight abuse of notation in this formula commited by applying the $\circ$ operation in a situation in which one of the sets is not invariant by double perpendicularity.

5. From the above Definition 3.4, we can deduce a crucial "half adjunction property" relating the operations $\circ$ and $\to$ in $\mathcal{P}_\perp(\Pi)$.

Theorem 3.6. [Half adjunction property] *Assume that* $P, Q, R \in \mathcal{P}_\perp(\Pi)$. *If* $Q \to R \subseteq P$, *then* $R \subseteq P \circ Q$.

Proof. The inclusion $Q \to R \subseteq P$ means that $({}^\perp({}^\perp Q \cdot R))^\perp \subseteq P$ and this is equivalent to ${}^\perp Q \cdot R \subseteq P$. Now, this implies that $R \subseteq \{\pi \in \Pi : {}^\perp Q.\pi \subseteq P\}$ that implies that $R \subseteq P \circ Q$. □

Observation 3.7. (1) We have used the following elementary fact: if $P, Q, R \in \mathcal{P}_\perp(\Pi)$. Then, $P \supseteq {}^\perp Q \cdot R$ if and only if $\{\pi \in \Pi : P \supseteq {}^\perp Q.\pi\} \supseteq R$.

(2) From the above comment it follows that if $({}^\perp\{\pi \in \Pi : P \supseteq {}^\perp Q.\pi\})^\perp = \{\pi \in \Pi : P \supseteq {}^\perp Q.\pi\}$ –i.e. if $\{\pi \in \Pi : P \supseteq {}^\perp Q.\pi\} \in \mathcal{P}_\perp(\Pi)$– then the conditions $P \circ Q \supseteq R$ and $P \supseteq Q \to R$ are equivalent.

(3) Along the proof of Theorem 3.6 we obtained the following fact: the inclusion $Q \to R \subseteq P$ is equivalent to $({}^\perp({}^\perp Q \cdot R))^\perp \subseteq P$ that is equivalent to ${}^\perp Q \cdot R \subseteq P$.

Using the above adjunction result –Theorem 3.6–in the case that $P = Q \to R$ we obtain the following Corollary.

Corollary 3.8. *For all* $R, Q \in \mathcal{P}_\perp(\Pi)$, *we have that* $R \subseteq (Q \to R) \circ Q$.

6. It is important to remark that in fact, the operations $\circ$ and $\to$ are not independent. Their close relationship is illustrated in the theorem that follows .

4. Abstract Krivine Structures.

7. In this section we complete the definition of a calculus in a realizability lattice to obtain the concept of *pre–Abstract Krivine Structure* abbreviated as $\mathcal{PAKS}$. For that, we introduce the usual application map for terms, a save map from stacks to terms, the combinators K, S, and a distinguished term cc that is a realizer of Peirce's law.

Definition 4.1. A *pre–Abstract Krivine Structure* consists of the following elements:

(1) A nonuple

$$(\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{app}, \mathrm{save}, \mathrm{push}, \mathrm{K}, \mathrm{S}, \mathrm{cc}),$$

where:

- (a) $(\Lambda, \Pi, \perp\!\!\!\perp)$ is an $\mathcal{RL}$.
- (b) $\mathrm{app} : \Lambda \times \Lambda \to \Lambda$ is a function: $(t, u) \mapsto \mathrm{app}(t, u) = tu$.
- (c) $\mathrm{save} : \Pi \to \Lambda$ is a function: $\pi \mapsto \mathrm{save}(\pi) = k_\pi$.
- (d) $\mathrm{push} : \Lambda \times \Pi \to \Pi$ is a function. We abbreviate $(t, \pi) \mapsto \mathrm{push}(t, \pi) = t.\pi$.
- (e) $\mathrm{K}, \mathrm{S}, \mathrm{cc} \in \Lambda$ are distinguished elements.

The elements of $\Lambda \times \Pi$ are called *processes* and we denote the process (t, π) as $t \star \pi$.

(2) The above elements are subject to the following axioms.

- (S1) If $t \star s.\pi \in \perp\!\!\!\perp$, then $ts \star \pi \in \perp\!\!\!\perp$ –in the case that the converse holds, i.e. if $ts \star \pi \in \perp\!\!\!\perp$ implies that $t \star s \cdot \pi \in \perp\!\!\!\perp$, we say that the given $\mathcal{PAKS}$ is *strong*.
- (S2) If $t \star \pi \in \perp\!\!\!\perp$, then for all $s \in \Lambda$ we have that $\mathrm{K} \star t \cdot s \cdot \pi \in \perp\!\!\!\perp$.
- (S3) If $tu(su) \star \pi \in \perp\!\!\!\perp$, then $\mathrm{S} \star t \cdot s \cdot u \cdot \pi \in \perp\!\!\!\perp$.
- (S4) If $t \star k_\pi \cdot \pi \in \perp\!\!\!\perp$, then $\mathrm{cc} \star t \cdot \pi \in \perp\!\!\!\perp$.
- (S5) If $t \star \pi \in \perp\!\!\!\perp$, then for all $\pi' \in \Pi$ we have that $k_\pi \star t \cdot \pi' \in \perp\!\!\!\perp$.

8. Here and in the rest of these notes, product–like operations will –in general– be non associative. Hence, when parenthesis are omitted it is implicit that we associate to the left. In other words:

$$a_1a_2a_3 = (a_1a_2)a_3 \text{ and in general } a_1a_2a_3 \cdots a_n = (a_1a_2a_3 \cdots a_{n-1})a_n.$$

9. Notice, that besides adding the application, the save map and three distinguished terms to the structure of a *realizability lattice with a push*, we have introduced five axioms that interrelate the above data and that can be divided into three groups. The first axiom interrelates the newly defined application map with the push. The second and third establishes interactions between the combinators and the push map, while the fourth and fifth establishes relations between the push map, the save map and the distinguished element cc.

10. The elements of the structure above, named as:

$$save : \pi \mapsto k_\pi : \Pi \to \Lambda \quad \text{and} \quad \mathrm{cc} \in \Lambda,$$

have a very special role in the sense that they make the realizability theory *classical* as cc realizes Pierce's law. In this sense it may be convenient to introduce the following nomenclature, in the presence of the mentioned elements and the corresponding axioms (S4) and (S5), we say that the $\mathcal{PAKS}$ –and later the $\mathcal{AKS}$– is *classical*.

11. The axioms for a $\mathcal{PAKS}$ appearing in Definition 4.1, (2) can be formulated also as follows:

(S1) If $t \perp s \cdot \pi$, then $ts \perp \pi$ –moreover $ts \perp \pi$ if and only if $t \perp s \cdot \pi$ in the case that the given $\mathcal{PAKS}$ is strong.
(S2) If $t \perp \pi$, then for all $s \in \Lambda$ we have that $\mathrm{K} \perp t \cdot s \cdot \pi$.
(S3) If $tu(su) \perp \pi$, then $\mathrm{S} \perp t \cdot s \cdot u \cdot \pi$.
(S4) If $t \perp k_\pi \cdot \pi$, then $\mathrm{cc} \perp t \cdot \pi$.
(S5) If $t \perp \pi$, then for all $\pi' \in \Pi$ we have that $k_\pi \perp t \cdot \pi'$.

Observation 4.2. The following weaker consequences of the last three axioms can be deduced easily applying (S1) to (S2)...(S5). In the case that the $\mathcal{PAKS}$ is strong, the conditions below are equivalent to the original ones.

(1) If $t \perp \pi$, then for all $s \in \Lambda$ we have that $\mathrm{K}\,ts \perp \pi$.
(2) If $tu(su) \perp \pi$, then $\mathrm{S}\,tsu \perp \pi$.
(3) If $t \perp k_\pi \cdot \pi$, then $\mathrm{cc}\,t \perp \pi$. If the $\mathcal{PAKS}$ is strong we have that: if $tk_\pi \perp \pi$, then $\mathrm{cc}\,t \perp \pi$.
(4) If $t \perp \pi$, then for all $\pi' \in \Pi$ we have that $k_\pi t \perp \pi'$. In other words if $t \perp \pi$, then $k_\pi t \in {}^\perp\Pi$.

Observe that the last assertion exhibits a situation related to Observation 2.3, (4)

12. Once we have at our disposal the map app : $(t, s) \mapsto ts : \Lambda \times \Lambda \to \Lambda$, we can define the following *conductor* for $L, M \subseteq \Lambda$ –compare with the previous Definition 3.2–:

$$L \rightsquigarrow M = \{t \in \Lambda : tL \subseteq M\} \subseteq \Lambda \quad \text{left conductor of } L \text{ into } M.$$

Notice that similarly than before, the above conductor can be characterized in the following way:

$$L \rightsquigarrow M = \bigcup\{L' \subseteq \Lambda : L'L \subseteq M.\}$$

Considering also the natural operator coming from the application –app–:

$$(L, M) \mapsto LM : \mathcal{P}(\Lambda) \times \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda),$$

we may define the maps $a_L, m_L : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$ by $a_L(M) = L \rightsquigarrow M$ and $m_L(M) = LM$[3].

We have the following adjoint relationship: for all $L, M, N \subseteq \Lambda$;

$$m_L(N) \subseteq M \Leftrightarrow N \subseteq a_L(M).$$

Definition 4.3. For $P, Q \in \mathcal{P}_\perp(\Pi)$ we define the following binary operation in $\mathcal{P}_\perp(\Pi)$:

$$P \diamond Q = \mathrm{app}({}^\perp P, {}^\perp Q)^\perp = (({}^\perp P)({}^\perp Q))^\perp \in \mathcal{P}_\perp(\Pi).$$

Observation 4.4. The importance of the three operations $\circ$, $\to$ and $\diamond$ defined in $\mathcal{P}_\perp(\Pi)$ can be visualized when one performs the following computations in singleton sets.

[3] In principle, the maps defined above are not internal maps in the corresponding $\mathcal{P}_\perp$s.

(1) Let us take $t, s \in \Lambda$. We have that:

$$\{t\}^\perp \circ \{s\}^\perp = ({}^\perp\{\pi \in \Pi : r \perp \ell.\pi \ \forall r \perp \{t\}^\perp, \forall \ell \perp \{s\}^\perp\})^\perp, \tag{4.4.1}$$

and as $t \in \{t\}^\perp$; $s \in \{s\}^\perp$ we deduce that: $({}^\perp\{\pi \in \Pi : r \perp \ell.\pi \ \forall r \perp \{t\}^\perp, \forall \ell \perp \{s\}^\perp\})^\perp \subseteq ({}^\perp\{\pi \in \Pi : t \perp s.\pi\})^\perp$. Then:

$$\{t\}^\perp \circ \{s\}^\perp \subseteq ({}^\perp\{\pi \in \Pi : t \perp s.\pi\})^\perp. \tag{4.4.2}$$

(2) Moreover, by definition we have that $\{t\}^\perp \diamond \{s\}^\perp = \left(({}^\perp(\{t\}^\perp))({}^\perp(\{s\}^\perp))\right)^\perp$ and:

$$\{t\}^\perp \diamond \{s\}^\perp = \{\pi \in \Pi : r\ell \perp \pi \ \forall r \perp \{t\}^\perp, \forall \ell \perp \{s\}^\perp\}. \tag{4.4.3}$$

Hence:

$$\{t\}^\perp \diamond \{s\}^\perp \subseteq \{ts\}^\perp. \tag{4.4.4}$$

(3) Next we show that there is a very close relationship between the operations $\circ, \diamond$ and the basic condition (S1) of Definition 4.1, (2).
Indeed, condition (S1) implies that:

$$\begin{aligned}({}^\perp\{\pi \in \Pi : r \perp \ell.\pi \ \forall r \perp \{t\}^\perp, \forall \ell \perp \{s\}^\perp\})^\perp &\subseteq ({}^\perp\{\pi \in \Pi : r\ell \perp \pi \ \forall r \perp \{t\}^\perp, \forall \ell \perp \{s\}^\perp\})^\perp = \\ &= \{\pi \in \Pi : r\ell \perp \pi \ \forall r \perp \{t\}^\perp, \forall \ell \perp \{s\}^\perp\}.\end{aligned}$$

Using the characterization of the operations appearing in (1) and (4.4.3), we deduce that in the presence of condition (S1) we have that for all t, s

$$\{t\}^\perp \circ \{s\}^\perp \subset \{t\}^\perp \diamond \{s\}^\perp.$$

(4) Concerning the implication we have:

$$\{t\}^\perp \to Q = ({}^\perp({}^\perp(\{t\}^\perp).Q))^\perp \supseteq ({}^\perp(t.Q))^\perp,$$

or equivalently:

$${}^\perp(\{t\}^\perp \to Q) \subseteq {}^\perp(t.Q).$$

13. For future use, it is interesting to write down the basic axioms of a $\mathcal{PAKS}$ in terms of elements of $\mathcal{P}_\perp(\Lambda)$ and $\mathcal{P}_\perp(\Pi)$ and the operations $\circ$, $\to$, $\diamond$ and the conductors. We emphasize –with an eye in future use– the formulation in terms of $\mathcal{P}_\perp(\Pi)$.

Lemma 4.5. *The axioms of a $\mathcal{PAKS}$ presented in Definition 4.1,* (2) *–also appearing in an equivalent formulation in* **11.**–*, have the following consequences. Assume that P, Q, R, are generic elements of $\mathcal{P}_\perp(\Pi)$, and that $\mathrm{K}, \mathrm{S}, \mathrm{cc} \in \Lambda$ are as before, then:*

($\mathbb{S}1$) *Condition* (S1) *in Definition 4.1,* (2) *implies condition* (1) *that implies condition* (3) *that implies* (2).
  (1) *$P \circ Q \subseteq ({}^\perp P {}^\perp Q)^\perp = P \diamond Q$ or equivalently ${}^\perp P {}^\perp Q \subseteq {}^\perp(P \circ Q)$ or equivalently: if $t \perp P$ and $s \perp Q$, then $ts \perp P \circ Q$.*
  (2) *$({}^\perp P \rightsquigarrow {}^\perp Q)^\perp \subseteq P \to Q$ or equivalently $Q \subseteq (P \to Q) \diamond P$.*
  (3) *If ${}^\perp Q.R \subseteq P$, then ${}^\perp P {}^\perp Q \subseteq {}^\perp R$. Equivalently, if $Q \to R \subseteq P$, then $R \subseteq P \diamond Q$.*

($\mathbb{S}2$) *The first condition below is equivalent to condition* (S2) *in Definition 4.1,* (2)*, and the second is a consequence.*
  (1) *For all P, R, we have that $\mathrm{K} \in {}^\perp({}^\perp P.{}^\perp R.P)$. Equivalently, for all $P \subseteq Q$ we have that $K \in {}^\perp({}^\perp Q.{}^\perp R.P)$.*
  (2) *For all P, R, we have that $\mathrm{K}\,{}^\perp P {}^\perp R \subseteq {}^\perp P$. Equivalently, for all $P \subseteq Q$ we have that $\mathrm{K}\,{}^\perp Q {}^\perp R \subseteq {}^\perp P$*

($\mathbb{S}3$) *The first condition below is equivalent to condition* (S3) *in Definition 4.1,* (2)*, and the second is a consequence.*
  (1) *If ${}^\perp Pu({}^\perp Qu) \subseteq {}^\perp R$ then $\mathrm{S} \in {}^\perp({}^\perp P.{}^\perp Q.u.R)$ with $u \in \Lambda$.*
  (2) *If ${}^\perp Pu({}^\perp Qu) \subseteq {}^\perp R$ then $\mathrm{S}\,{}^\perp P {}^\perp Qu \subseteq {}^\perp R$ with $u \in \Lambda$.*

($\mathbb{S}4$) (1) *Axiom* (S4) *in Definition 4.1 is equivalent to:* $\mathrm{cc} \in {}^\perp((\mathrm{save}(P).P) \to P)$.
  (2) *Axiom* (S5) *in Definition 4.1 is equivalent to:* $\mathrm{save}(P) \subseteq {}^\perp(P \to Q)$.

(3) *Axioms* (S4) *and* (S5) *imply that for all* $P, Q \in \mathcal{P}_\perp(\Pi)$*:* $\mathrm{cc} \perp ((P \to Q) \to P) \to P$. *In other words the axioms imply that the term* $\mathrm{cc} \in \Lambda$ *realises Peirce's law.*

Proof. ($\mathbb{S}1$)

- It is evident that the three formulations of condition (1) are equivalent.
- The two formulations of condition (2) are equivalent. Indeed, for arbitrary $L, M \subseteq \Lambda$ we have that $(L \rightsquigarrow M) = \bigcup\{N \subseteq \Lambda : NL \subseteq M\}$ and then $(L \rightsquigarrow M)^\perp = \bigcap\{N^\perp \subseteq \Pi : NL \subseteq M\}$ and then $({}^\perp P \rightsquigarrow {}^\perp Q)^\perp = \bigcap\{N^\perp \subseteq \Pi : N({}^\perp P) \subseteq {}^\perp Q\}$. Call $N_0 = {}^\perp(P \to Q)$, in accordance to the above equality in order to prove that $({}^\perp P \rightsquigarrow {}^\perp Q)^\perp \subseteq (P \to Q) = N_0^\perp$, all we have to show is that $N_0({}^\perp P) \subseteq {}^\perp Q$ or in other words that ${}^\perp(P \to Q)({}^\perp P) \subseteq {}^\perp Q$. Taking perpendiculars in the above inequality we show that our statement implies that $Q \subseteq (P \to Q) \diamond P$.
  Conversely, the inclusion $Q \subseteq (P \to Q) \diamond P$ implies that ${}^\perp(P \to Q)({}^\perp P) \subseteq {}^\perp Q$, which in turn implies –by the definition of the conductor– that ${}^\perp(P \to Q) \subseteq {}^\perp P \rightsquigarrow {}^\perp Q$. Taking perpendiculars again in this inclusion we deduce that $({}^\perp P \rightsquigarrow {}^\perp Q)^\perp \subseteq (P \to Q)$.
- Also, the two formulations of condition (3) are equivalent. Indeed, it is clear that ${}^\perp Q.R \subseteq P$ if and only if $(Q \to R) = ({}^\perp({}^\perp Q.R))^\perp \subseteq P$ and also it follows that ${}^\perp P {}^\perp Q \subseteq {}^\perp R$ can also be written as $R = ({}^\perp R)^\perp \subseteq ({}^\perp P {}^\perp Q)^\perp = P \diamond Q$.
- Assuming that the original formulation of rule (S1) holds, we want to prove (1), which is the assertion that for all $P, Q \in \mathcal{P}_\perp(\Pi)$, then:
  $$\{\pi \in \Pi : {}^\perp Q.\pi \subseteq P\} \subseteq ({}^\perp P {}^\perp Q)^\perp.$$
  In other words we want to show that if $\pi \in \Pi$ is such that ${}^\perp Q.\pi \subseteq P$ then, for all $s \perp P$, $t \perp Q$ we have that $st \perp \pi$. It is clear that from the hypothesis ${}^\perp Q.\pi \subseteq P$ and $s \perp P$, $t \perp Q$, that $s \perp t.\pi$ and in this case the original condition (S1) implies that $st \perp \pi$.
- Now we prove that condition (1) implies condition (3). Using the "half adjunction property" from the hypothesis of (3): $(Q \to R) \subseteq P$ we deduce that $R \subseteq P \circ Q$ and using (1) we prove that $R \subseteq P \circ Q \subseteq P \diamond Q$.
- Next we prove that (3) implies (2). Consider the equality $(P \to Q) = (P \to Q)$ and using (3) deduce that $Q \subseteq (P \to Q) \diamond P$ that is exactly the statement of (2).

($\mathbb{S}2$) Observe that both versions of condition (1) are equivalent. We prove first that our condition (1) implies the original condition (S2). Assume that $t \perp \pi$, we want to show that for all $s \in \Lambda$ we have that $K \perp (t.s.\pi)$. Call $P = ({}^\perp\{\pi\})^\perp$ and $R = \{s\}^\perp$. From the assertion that $K \perp {}^\perp P.{}^\perp R.P$ as $t \in {}^\perp P$, $s \in {}^\perp R$ and $\pi \in P$ we conclude that $K \perp (t.s.\pi)$.
Conversely, suppose the take the subset ${}^\perp P.{}^\perp R.P \subseteq \Pi$ and we want to prove that for all $t \perp P$, $s \perp R$ and $\pi \in P$, $K \perp t.s.\pi$. As $t \perp \pi$ from the original condition (S2) we deduce that $K \perp (t.s.\pi)$ that is exactly what we needed to prove. The fact that condition (1) implies condition (2) is a direct consequence of the axiom (S1) of a $\mathcal{PAKS}$.

($\mathbb{S}3$) The proof of this part uses the same methods than the previous one.

($\mathbb{S}4$) Axiom (S5) can be written as the assertion: $\mathrm{save}(P) \subseteq {}^\perp({}^\perp P.Q) = {}^\perp(P \to Q)$ for all P, Q and axiom (S4) can be written as the assertion: $\mathrm{cc} \in {}^\perp\Big({}^\perp(\mathrm{save}(P).P).P\Big) = {}^\perp\Big((\mathrm{save}(P).P) \to P\Big)$ for all P.
Putting this together, we obtain that:

$$\mathrm{cc} \in {}^\perp\Big((\mathrm{save}(P).P) \to P\Big) \subseteq {}^\perp\Big(({}^\perp(P \to Q).P) \to P\Big) \subseteq {}^\perp\Big(((P \to Q) \to P) \to P\Big) \text{ for all } P, Q \in \mathcal{P}_\perp(\Pi).$$

□

14. In accordance with Theorem 3.6 (*half adjunction property*) we have that: if $Q \to R \subseteq P$, then $R \subseteq P \circ Q$. In search of a version of a converse to this result–i.e to obtain the other "half" of the adjunction, we introduce the so called "E *operator*" and the associated "Sη *rule*".

Theorem 4.6. *In a* $\mathcal{PAKS}$ *if* $t, s \in \Lambda$ *we have that:*

$$ts \perp \pi \Rightarrow \mathrm{S(K(S\,K\,K))}t \perp s.\pi.$$

Proof. The proof is performed in two steps.

(1) If $t \perp \pi$, then $\mathrm{S\,K\,K} \perp t.\pi$. Indeed:

$$t \perp \pi \Rightarrow \mathrm{K} \perp t.(\mathrm{K}\,t).\pi \Rightarrow (\mathrm{K}\,t)(\mathrm{K}\,t) \perp \pi \Rightarrow \mathrm{S} \perp \mathrm{K}\,.\,\mathrm{K}\,.t.\pi \Rightarrow \mathrm{S\,K\,K} \perp t.\pi.$$

The validity of the succesive implications come by respective application of the following axioms **11.** (S2),(S1),(S3), and (S1) in that order.

(2) If $ts \perp \pi$, then $\mathrm{S(K(S\,K\,K))}t \perp s.\pi$. The following chain of implications proves the result:

$$ts \perp \pi \Rightarrow \mathrm{S\,K\,K} \perp ts.\pi \Rightarrow \mathrm{K} \perp \mathrm{SKK}\,.s.ts.\pi \Rightarrow \mathrm{K(SKK)}s(ts) \perp \pi$$
$$\mathrm{K(SKK)}s(ts) \perp \pi \Rightarrow \mathrm{S} \perp (\mathrm{K(SKK)}).t.s.\pi \Rightarrow \mathrm{S(K(SKK))}t \perp s.\pi.$$

The list of the axioms or results used at each respective implication is: Part (1) above, **11.** (S2), (S1), (S3) and (S1).

□

Definition 4.7. The special elements of Λ considered above are abbreviated as follows:

$$\mathrm{I} = \mathrm{SKK}\,;\ \mathrm{E} = \mathrm{S(K\,I)} = \mathrm{S(K(SKK))}.$$

Thus, the $\mathrm{S}\,\eta$ rule can be formulated as:

$$ts \perp \pi \Rightarrow \mathrm{E}\,t \perp s.\pi.$$

Next we present a set theoretical characterization of the $\mathrm{S}\,\eta$ rule that can be proved easily.

Lemma 4.8. (1) *A combinator $\widehat{\mathrm{E}}$ satisfies the $\mathrm{S}\,\eta$ rule –i.e. $ts \perp \pi \Rightarrow \widehat{\mathrm{E}}t \perp s.\pi$– if and only if satisfies any of the the assertions that follow.*

$$\text{If } P, Q \in \mathcal{P}_\perp(\Pi) \text{ then } P \diamond Q \subseteq \{\pi \in \Pi : \widehat{\mathrm{E}}^\perp P \subseteq {}^\perp({}^\perp Q.\pi)\} = \{\pi \in \Pi : (\widehat{\mathrm{E}}^\perp P)^\perp \supseteq ({}^\perp Q.\pi)\}. \tag{4.8.5}$$

$$\text{If } P, Q \in \mathcal{P}_\perp(\Pi) \text{ then } \widehat{\mathrm{E}}^\perp P \subseteq {}^\perp({}^\perp Q.(P \diamond Q)). \tag{4.8.6}$$

$$\text{If } R \subseteq (P \diamond Q), \text{ with } P, Q, R \in P_\perp(\Pi) \text{ then } \widehat{\mathrm{E}}^\perp P \subseteq {}^\perp({}^\perp Q.R).^4 \tag{4.8.7}$$

(2) *If the combinator $\widehat{\mathrm{E}}$ satisfies the $\mathrm{S}\,\eta$ rule then, the assertions that follow –see the notations of Definition 3.4– are valid.*

$$\text{If } P, Q \in \mathcal{P}_\perp(\Pi) \text{ then } \widehat{\mathrm{E}}({}^\perp({}^\perp P.Q)) \subseteq {}^\perp({}^\perp P.Q) \text{ or equivalently } \widehat{\mathrm{E}}({}^\perp(P \to Q)) \subseteq {}^\perp(P \to Q). \tag{4.8.8}$$

$$(t({}^\perp P))^\perp \subseteq \{\pi \in \Pi : (\widehat{\mathrm{E}}t)^\perp \supseteq ({}^\perp P.\pi)\} \subseteq ({}^\perp\{\pi \in \Pi : (\widehat{\mathrm{E}}t)^\perp \supseteq ({}^\perp P.\pi)\})^\perp = (\widehat{\mathrm{E}}t)^\perp \circ P. \tag{4.8.9}$$

$$\text{If } P, Q \in \mathcal{P}_\perp(\Pi) \text{ then } (P \diamond Q) \subseteq (\widehat{\mathrm{E}}({}^\perp P))^\perp \circ Q. \tag{4.8.10}$$

Proof. (1)
- It is clear that the assertions (4.8.5),(4.8.6),(4.8.7) are all equivalent.
- The inclusion (4.8.5) is equivalent to the assertion: $\forall s, t, \pi,\ ts \perp \pi \Rightarrow \widehat{\mathrm{E}}t \perp s.\pi$.
  Assume that $ts \perp \pi$ and call $P = \{t\}^\perp$ and $Q = \{s\}^\perp$. Clearly $ts \in {}^\perp P^\perp Q$ and then $\pi \in P \diamond Q$ and in the situation that the inclusion (4.8.5) is valid, we deduce that $\widehat{\mathrm{E}}({}^\perp P) \subseteq {}^\perp({}^\perp Q.\pi)$. As $t \in {}^\perp P$ and $s \in {}^\perp Q$, we obtain that $\widehat{\mathrm{E}}t \perp s.\pi$. The converse can be proved by reversing the above argument.

(2)
- For the proof of the fact the $\mathrm{S}\,\eta$ rule implies the inclusion (4.8.8) we proceed as follows.
  Assume that $t \in {}^\perp({}^\perp P.Q)$, then $t \perp s.\pi$ for all $s \in {}^\perp P$ and $\pi \in Q$. In this situation we deduce that $ts \perp \pi$ and applying the $\mathrm{S}\,\eta$ rule we deduce that $\widehat{\mathrm{E}}t \perp s.\pi$. This means that $\widehat{\mathrm{E}}t \in {}^\perp({}^\perp P.Q)$.
- Next we show that the the $\mathrm{S}\,\eta$ rule implies the inclusion (4.8.9).
  Assume as hypothesis the validity of the $\mathrm{S}\,\eta$ rule. Take $\pi \in (t({}^\perp P))^\perp$–i.e. assume that for all $s \perp P$, $ts \perp \pi$. Using the hypothesis we deduce that for all $s \perp P$ we have that $\widehat{\mathrm{E}}t \perp s.\pi$ and that means that ${}^\perp P.\pi \subseteq (\widehat{\mathrm{E}}t)^\perp$ and that implies that the inclusion (4.8.9) is valid.
- The validity of (4.8.10) is a consequence of the following chain of inclusions –the first one is just the inclusion (4.8.5)–:
  $$P \diamond Q \subseteq \{\pi \in \Pi : (\widehat{\mathrm{E}}({}^\perp P))^\perp \supseteq ({}^\perp Q.\pi)\} \subseteq ({}^\perp\{\pi \in \Pi : (\widehat{\mathrm{E}}({}^\perp P))^\perp \supseteq ({}^\perp Q.\pi)\})^\perp = (\widehat{\mathrm{E}}({}^\perp P))^\perp \circ Q.$$

[4] In terms of subsets of $\mathcal{P}_\perp(\Lambda)$ it can be formulated as: $LL' \subseteq M \in \mathcal{P}_\perp(\Lambda)$, then $\widehat{\mathrm{E}}L \subseteq {}^\perp(L'.M^\perp)$.

□

Corollary 4.9. *For all $P \in \mathcal{P}_\perp(\Pi)$ and for* E *as before, we have that:*

$$(E(^\perp P))^\perp \subseteq (\mathrm{EE})^\perp \circ P. \tag{4.9.11}$$

Proof. This assertion is a particular case of (4.8.9) when $t = \mathrm{E}$. □

Observation 4.10. (1) Another consequence of the S η rule, that follows directly from the above results –see inclusion (4.8.10), as well as Definition 3.4, (1)–is the following:

$$\text{If } P, Q \in \mathcal{P}_\perp(\Pi) \text{ then } P \diamond Q \subseteq \left({}^\perp({}^\perp Q \rightsquigarrow (\mathrm{E}({}^\perp P))^\perp)\right)^\perp. \tag{4.10.12}$$

(2) Notice that we have proved that the operator E contracts subsets of Λ of the form: ${}^\perp(P \to Q)$ for P and Q in the corresponding $\mathcal{P}_\perp(\Pi)$–see property (4.8.8). It does not seem possible to prove that E contracts all subsets L in $\mathcal{P}_\perp(\Lambda)$.

(3) If we put together the above equation (4.8.10) and Lemma 4.5, ($\mathbb{S}1$), (1), we obtain:

$$P \circ Q \subseteq P \diamond Q \subseteq (\mathrm{E}({}^\perp P))^\perp \circ Q. \tag{4.10.13}$$

The theorem that follows –that is of importance for future developments–is a partial converse to the half adjunction property of Theorem 3.6.

Theorem 4.11. *Let $P, Q, R \in \mathcal{P}_\perp(\Pi)$. If $P \circ Q \supseteq R$ then $\mathrm{E}\,{}^\perp P \subseteq {}^\perp(Q \to R)$. Equivalently, if $P \circ Q \supseteq R$ then $(\mathrm{E}\,{}^\perp P)^\perp \supseteq (Q \to R)$.*

Proof. As $R \subseteq P \circ Q \subseteq P \diamond Q = ({}^\perp P {}^\perp Q)^\perp$ –see Lemma 4.5 (S1) (1)–, we have that ${}^\perp Q.R \subseteq {}^\perp Q.({}^\perp P {}^\perp Q)^\perp$ and ${}^\perp({}^\perp Q.R) \supseteq {}^\perp({}^\perp Q.({}^\perp P {}^\perp Q)^\perp)$. Using the inclusion (4.8.6) we deduce that $\mathrm{E}\,{}^\perp P \subseteq {}^\perp({}^\perp Q.({}^\perp P {}^\perp Q)^\perp) \subseteq {}^\perp({}^\perp Q.R) = {}^\perp(Q \to R)$ that is the inequality we wanted to prove.

Clearly the inequality $\mathrm{E}\,{}^\perp P \subseteq {}^\perp(Q \to R)$ is equivalent to $(\mathrm{E}\,{}^\perp P)^\perp \supseteq (Q \to R)$ –see Observation 2.6–. □

15. In order to summarize, we write down explicitly the adjunction properties valid in a general $\mathcal{PKAS}$. We also put them together –for future use– with the conclusion of (4.9.11).

Theorem 4.12. *Assume that $P, Q, R \in \mathcal{P}_\perp(\Pi)$.*

$$(Q \to R) \subseteq P \quad \Rightarrow \quad R \subseteq P \circ Q \tag{4.12.14}$$

$$R \subseteq P \circ Q \quad \Rightarrow \quad (Q \to R) \subseteq (\mathrm{E}\,{}^\perp P)^\perp \subseteq (\mathrm{EE})^\perp \circ P \tag{4.12.15}$$

16. When we add to the $\mathcal{PAKS}$ a subset of terms called quasi proofs we obtain the concept of *Abstract Krivine Structure* –$\mathcal{AKS}$. This last concept was introduced by J.L. Krivine and generalized by T. Streicher –see [7] and [9] respectively–.

Definition 4.13. An *Abstract Krivine Structure* is a decuple:

$$(\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{app}, \mathrm{save}, \mathrm{push}, \mathrm{K}, \mathrm{S}, \mathrm{cc}, \mathrm{QP}),$$

where the nonuple:

$$(\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{app}, \mathrm{save}, \mathrm{push}, \mathrm{K}, \mathrm{S}, \mathrm{cc}),$$

is a $\mathcal{PAKS}$ and the subset $\mathrm{QP} \subseteq \Lambda$ whose elements are called *quasi proofs* satisfies the following conditions:

(Si) $\mathrm{K}, \mathrm{S}, \mathrm{cc} \in \mathrm{QP}$

(Sii) $\mathrm{app}(\mathrm{QP}, \mathrm{QP}) \subseteq \mathrm{QP}$.

Observation 4.14. It is clear that if QP is as in Definition 4.13, then E as well as EE are elements of QP.

17. The abbreviations and notations introduced along this section, will be in force in this notes.

5. Combinatory algebras and ordered combinatory algebras.

18. We recall the definition of combinatory algebra –abbreviated as $\mathcal{CA}$–.

Definition 5.1. A combinatory algebra is a quadruple $(A, \circ, \mathrm{k}, \mathrm{s})$ where A is a set, $\mathrm{k}, \mathrm{s} \in A$ is a pair of distinguished elements of A and $\circ : A \times A \to A$ is an operation –written as $\circ(a, b) = ab$ and called the application of A. The data displayed above are subject to the axioms: $\mathrm{k}ab = a$ and $\mathrm{s}abc = ac(bc)$.

Observation 5.2. The application taken above, is not necessarily associative, hence as it is customary we associate to the left: $abc = (ab)c$, etc.

The axioms introduced in Definition 5.1 mean:
(1) $(\mathrm{k}a)b = a$;
(2) $((\mathrm{s}a)b)c = (ac)(bc)$.

19. We perform some manipulations in a $\mathcal{CA}$.

Observation 5.3. (1) $\mathrm{skk}\,a = \mathrm{k}\,a(\mathrm{k}\,a) = a$, in other words the element skk behaves as the identity with respect to the operation in A.
The first equality is a direct consecuence of the second axiom of a combinatory algebra and the second equality follows directly from the first –see Definition 5.1–.
(2) $\mathrm{k}\,ab = a$, so that k works as the projection in the first coordinate.
(3) $\mathrm{k(skk)}\,ab = (\mathrm{k(skk)}\,a)b = \mathrm{skk}\,b = b$, so that k(skk) operates as the projection in the second coordinate.
The first equality is just the law of parenthesis, the second is the first axiom of a combinatory algebra and the third was just proved.

20. Next we recall the manner in which λ–calculus can be reformulated in the above framework without performing substitutions when using reduction.

Definition 5.4. Assume that we have $\mathcal{V}$ a countable set of variables that we denote as $x_1, x_2, \cdots$. Consider $\mathcal{U} \subseteq \mathcal{V}$ and define $A[\mathcal{U}]$ as the smallest set containing $\mathcal{U}$, k, s and that is closed under application.

Observe that each element of $A[\mathcal{V}]$ contains only a *finite* number of variables and then

$$A[\mathcal{V}] = \bigcup \{A[x_1, \ldots, x_k] \mid k \in \mathbb{N}\}$$

Theorem 5.5. *There is a function* $\lambda^* y : A[x_1, \cdots, x_k, y] \to A[x_1, \cdots, x_k]$ *satisfying the following property:*

$$\forall t \in A[x_1, \cdots, x_k, y]\ ,\ \forall u \in A[x_1, \cdots, x_k] \quad \text{then} \quad (\lambda^* y(t)) \circ u = t\{y := u\}.$$

Proof. We abbreviate $(\lambda^* y(t)) \circ u$ as $(\lambda^* y(t))u$. Denote $\lambda^* y(t) = \lambda^* y.t$. Define:
(1) $\lambda^* y.t = \mathrm{k}\,t$ provided that y does not appear in t.
(2) $\lambda^* y.y = \mathrm{skk}$.
(3) $\lambda^* y.(tu) = \mathrm{s}(\lambda^* y.t)(\lambda^* y.u)$.

□

21. Taking the above into account, one could define the standard Krivine abstract machine –abbreviated as $\mathcal{KAM}$– in the following manner.

Definition 5.6. (1) The terms and stacks are:

$$\Lambda : x \,|\, \mathrm{K} \,|\, \mathrm{S} \,|\, \mathrm{cc} \,|\, \mathrm{k}_\pi \,|\, ts \quad ; \quad \Pi : \alpha \,|\, t.\pi,$$

and as before the elements of the set Λ are called the terms and the elements of the set Π are called the stacks. The element α is called a constant stack.
As before, the elements of $\Lambda \times \Pi$ are called processes and a generic process is denoted as $t \star \pi$.
(2) The reduction is defined by the following rules:
(R1) $ts \star \pi \succ t \star s.\pi$;
(R2) $\mathrm{K} \star t.s.\pi \succ t \star \pi$;
(R3) $\mathrm{S} \star t.s.u.\pi \succ tu(su) \star \pi$;

(R4) $\mathrm{cc} \star t.\pi \succ t \star k_\pi.\pi$;
(R5) $\mathrm{k}_\pi \star t.\pi' \succ t \star \pi$.

Observation 5.7. It is worth noticing that the reduction rules introduced in Definition 5.6 are equivalent to the assertion that $\perp\!\!\!\perp$ is closed by the antireduction determined by the rules written in Definition 4.1 item (2).

Question 5.8. What are the differences between choosing as models one or the other of the following two contexts?

$$\mathcal{PAKS} \Leftrightarrow \mathcal{KAM}$$

Partial answer:

	$\mathcal{PAKS}$	$\mathcal{KAM}$
The processes	Λ, Π are more general.	Λ, Π are more standard.
The calculus	Can be more abstract.	Is more rigid.

For example, in a general $\mathcal{PAKS}$ the sets Λ and Π could be the same. Moreover, in the situation of an abstract $\mathcal{PAKS}$ the application can have properties that the standard λ–calculus does not have, e.g. it can be commutative.

22. As the definition of a $\mathcal{PAKS}$ does not involve an equality defined in advance, in order to relate this concept with the concept of a combinatory algebra, we need to relax the definitions and look at *ordered combinatory algebras* [2].

Definition 5.9. An ordered combinatory algebra –$\mathcal{OCA}$– consists of the following:

(1) A quintuple
$$(A, \circ, \leq, \mathrm{k}, \mathrm{s}),$$
where:
  (a) A is a set.
  (b) $\circ : A \times A \to A$ is a function $(a, b) \mapsto \circ(a, b) = a \circ b$ –the function $\circ$ is called the *application* and concerning this application we always associate to the left–.
  (c) The relation $\leq$ is a partial orden in A[5].
  (d) k and s are a pair of distinguished elements of A.
(2) The above ingredients are subject to the following axioms.
  (a) The map $\circ : A \times A \to A$ is monotone with respect to the cartesian product order in $A \times A$ –i.e. if $a \leq a'$ and $b \leq b'$, then $ab \leq a'b'$–.
  (b) The distinguished elements satisfy:
    (i) $\mathrm{k}ab \leq a$;
    (ii) $sabc \leq ac(bc)$.
(3) We say that the $\mathcal{OCA}$ is equipped with an implication, if there is a binary operation –called implication– $\to: A \times A \to A$ with the following properties:
  (a) (*Half Adjunction property*.) For all $a, b, c \in A$, if $a \leq (b \to c)$ then $ab \leq c$.
  (b) The map $\to: A \times A \to A$ is monotone in the second variable and antimonotone in the first.
(4) (*Adjunction property*.) We say that the $\mathcal{OCA}$ with implication has the *complete adjunction property* or simply the *adjunction property* if there is a distinguished element $\mathrm{e} \in A$, with the property that for all $a, b, c \in A$, if $ab \leq c$ then $\mathrm{e}\,a \leq (b \to c)$. The element e is called an *adjunctor*.
(5) We say that the $\mathcal{OCA}$ A is *classic*, if there is an element c with the property that $\mathrm{c} \leq (((a \to b) \to a) \to a)$.
(6) A subset $B \subset A$ is a sub–$\mathcal{OCA}$ if:
  (a) $\circ(B \times B) \subseteq B$.
  (b) $\mathrm{k}, \mathrm{s} \in B$.
  (c) If the original $\mathcal{OCA}$ has an implication $\to$, we ask B to satisfy that $\to(B \times B) \subseteq B$.
  (d) In the situation that A has an adjunctor $\mathrm{e} \in A$, we assume that $\mathrm{e} \in B$.

[5] Recall that a partial order in A is a relation $\leq\, \subseteq A \times A$ with the following properties: (1) Reflexivity: $a \leq a$; (2) Antisymmetry: $a \leq b$, $b \leq a$ implies, $a = b$; (3) Transitivity: $a \leq b$ and $b \leq c$, imply $a \leq c$. A partial order that do not necessarily satisfies (2), is called a preorder.

Observation 5.10. (1) It is clear that if $B \subseteq A$ is a sub–$\mathcal{OCA}$, then $(B, \circ|_{B\times B}, \leq|_{B\times B}, \mathrm{k}, \mathrm{s})$ is also an $\mathcal{OCA}$. Moreover, if A has an implication $\to$, then the restriction $\to|_{B\times B}$ is an implication for $(B, \circ|_{B\times B}, \leq|_{B\times B}, \mathrm{k}, \mathrm{s})$. Similarly, if e is an adjunctor for A that belongs to B, it is also an adjunctor for B.

(2) The property above –Definition 5.9, (3a) is called "half adjunction property", because of the following. If we fix $x \in A$, the morphisms $R_x : y \mapsto (x \to y) : A \to A$ and $L_x : y \mapsto (x \circ y) : A \to A$, satisfy the property that $a \leq R_b(c)$ implies that $L_b(a) \leq c$. If we view the preorder set $(A, \leq)$ as a category and the maps L_x, R_x as functors, the equivalence $a \leq R_b(c)$ if and only if $L_b(a) \leq c$ can be stated as: for all $x \in A$ the functor R_x is the right adjoint of L_x.

(3) In case that the original $\mathcal{OCA}$ has an adjunctor, we have the following situation: for all $a, b, c \in A$:

$$a \leq R_b(c) \Rightarrow L_a(b) \leq c \Rightarrow \mathrm{e}\, a \leq R_b(c).$$

Definition 5.11. Assume that in (the $\mathcal{OCA}$) A, we have a subset $X \subseteq A$. Define the sub–$\mathcal{OCA}$, $\langle X\rangle = \bigcap\{B \subseteq A : X \subseteq B, B \text{ sub } \mathcal{OCA} \text{ of } A\}$.

In the case A has an adjunctor, we assume that $\mathcal{OCA}$s we take in the intersection always contain e. This is in order to guarantee that $\langle X\rangle$ has an adjunctor.

Observation 5.12. It is important to remark the following difference. In combinatory algebras the concept or reduction is not present, only the concept of computation. In the present context, the symbol $\leq$ should be interpreted as "reduces to".

23. We perform some computations in the $\mathcal{OCA}$.

Lemma 5.13. *If A is an $\mathcal{OCA}$, the following properties are valid.*

(1) *If for $b \in A$ we call* $\mathrm{i}_b = \mathrm{sk}\, b$ *we have that* $\mathrm{i}_b\, a \leq a$ *for all $a \in A$. In particular the same is valid for* $\mathrm{i} = \mathrm{i}_\mathrm{k}$

(2) $\mathrm{ki}\, a = \mathrm{k(skk)}\, a \leq \mathrm{skk}$ *and* $\mathrm{ki}\, ab = \mathrm{k(skk)}\, ab \leq b$.

(3) *Call* $\mathrm{e}_0 = \mathrm{s(ki)}$, *then* $\mathrm{e}_0\, ab \leq ab$.

(4) *In particular* $\mathrm{e}_0\, \mathrm{e}_0\, a \leq \mathrm{e}_0\, a$.

Proof. (1) We have that: $(((\mathrm{sk})b)a) \leq (\mathrm{k}\, a)(ba) \leq a$, using the conditions appearing in Definition 5.9, (2b).

(2) We prove the second inequality, the first is similar: $\mathrm{k(skk)}\, ab = (\mathrm{k(skk)}\, a)b \leq \mathrm{skk}\, b \leq b$.

(3) $\mathrm{e}_0\, ab = \mathrm{s(ki)}\, ab \leq (\mathrm{ki}\, b)(ab) \leq \mathrm{i}(ab) \leq ab$.

(4) The inequality $\mathrm{e}_0\, \mathrm{e}_0\, a \leq \mathrm{e}_0\, a$, follows directly from the previous result.

□

24. Let A be an $\mathcal{OCA}$, we introduce the concept of filter in A.

Definition 5.14. A subset $\Phi \subseteq A$ is said to be a filter if:

(F1) The subset Φ is closed under application.
(F2) $\mathrm{k}, \mathrm{s} \in \Phi$.
(F3) If A has an adjunctor e, then $\mathrm{e} \in \Phi$.
(F4) If A is classic, we assume that $c \in \Phi$.

Observation 5.15. It is clear that given A and Φ as above, if we restrict to the filter the application and the order, then Φ becomes a sub $\mathcal{OCA}$ of A.

25. In what follows, we will program directly in the $\mathcal{OCA}$, using the standard codifications in the combinatory algebras.

Definition 5.16. Let A be an $\mathcal{OCA}$ and take a countable set of *variables*: $\mathcal{V} = \{x_1, x_2, \cdots\}$. Consider $A(\mathcal{V})$ –called *the set of terms in A*– that is the set of formal expressions given by the following grammar:

$$p_1, p_2 ::= a \quad | \quad x \quad | \quad p_1 p_2$$

where $a \in A$ and $x \in \mathcal{V}$. We denote as $A(x_1, \ldots, x_k)$ the set of terms in A containing only the variables $x_1, \cdots, x_k$. The term $p_1 p_2$ is called the application of p_1 and p_2.

We can endow canonically a quotient of $A(\mathcal{V})$ with an $\mathcal{OCA}$ structure.

Observation 5.17. Consider the –minimal– partial preorder R on $A(\mathcal{V})$ defined by the following statements:

(1) For $a, b \in A$ and if $a \leq b$, then $a \; R \; b$.
(2) For all $a, b \in A$: $ab \; R \; a \circ b$ and $a \circ b \; R \; ab$.
(3) If $p_1, p_2, q_1, q_2 \in A(\mathcal{V})$ are such that $p_1 \; R \; p_2$ and $q_1 \; R \; q_2$ then $p_1 q_1 \; R \; p_2 q_2$.
(4) If $p_1, p_2 \in A(\mathcal{V})$ then $\mathrm{k}\, p_1 p_2 \; R \; p_1$.
(5) If $p_1, p_2, p_3 \in A(\mathcal{V})$ then $\mathrm{s}\, p_1 p_2 p_3 \; R \; p_1 p_3 (p_2 p_3)$.

Notice that this minimal preorder exists because we can take the intersection of the non empty family of preorders that satisfy the above conditions and the family is not empty because it always contains the trivial relation $A(\mathcal{V}) \times A(\mathcal{V})$.

Define an equivalence relation $\equiv_R$ on $A(\mathcal{V})$ as: $p \equiv_R q$ iff $p \; R \; q$ and $q \; R \; p$. Thus, the order R can be factored to the quotient $A[\mathcal{V}] := A(\mathcal{V})/ \equiv_R$ endowing it with a partial order. This quotient is called the set of polynomials in A. Observe that $ab \equiv_R a \circ b$ for all $a, b \in A$.

In order to simplify notations, we will use the same symbol p to denote a polynomial (an element of the quotient) as well as for a term which belongs to the equivalence class of p.

Observe that in accordance with (3), if p_1, p_1', p_2, p_2' are terms such that $p_1 \equiv_R p_1'$ and $p_2 \equiv_R p_2'$ then $p_1 p_2 \equiv_R p_1' p_2'$. Thus the application of terms induces a corresponding "application" of polynomials that we denote as $p_1 \star p_2$.

Then, by definition, $(A[\mathcal{V}], R, \star)$ is an $\mathcal{OCA}$ and $(A, \leq, \circ)$ is a sub-$\mathcal{OCA}$ of $(A[\mathcal{V}], R, \star)$.

Abusing slightly the notations and when there are not possibilities of confusion, we denote the relation R as $\leq$ and the operation $\star$ as $\circ$ or as the concatenation of the factors. Also we call the elements of $A[\mathcal{V}]$ *terms* instead of *polynomials*. We say that $(A[\mathcal{V}], \leq, \circ)$ is an extension of $(A, \leq, \circ)$.

Theorem 5.18. *For any finite set of variables $\{x_1, \cdots, x_k, y\}$, there is a function $\lambda^* y : A[x_1, \cdots, x_k, y] \to A[x_1, \cdots, x_k]$ satisfying the following property:*

$$\textit{If } t \in A[x_1, \cdots, x_k, y]\ , \textit{ and } u \in A[x_1, \cdots, x_k] \quad \text{then} \quad (\lambda^* y(t)) \circ u \leq t\{y := u\}. \tag{5.18.16}$$

Moreover if $X \subseteq A$ is an arbitrary subset and t is a term with all its coefficients in X, then $\lambda^* y(t)$ is a term with all its coefficients in $\langle X \rangle$. In particular if all the coefficients of t are in the filter Φ, then $\lambda^* y(t)$ is a polynomial with all the coefficients in Φ.

Proof. We give the following recursive definition for $\lambda^* y$:

- If y does not appears in t, then $\lambda^* y(t) := \mathrm{k}\, t$
- $\lambda^* y(y) := \mathrm{skk}$
- If p, q are polynomials in $A[x_1, \ldots, x_k, y]$, then $\lambda^* y(pq) := \mathrm{s}(\lambda^* y(p))(\lambda^* y(q))$

Next we show that this function satisfies the requirements. Let us consider $u \in A[x_1, \ldots, x_k]$. If $x \neq y$ then $(\lambda^* y(x))u = \mathrm{k}\, xu \leq x = x\{y := u\}$. $(\lambda^* y(y))u = \mathrm{skk}\, u \leq \mathrm{k}\, u(\mathrm{k}\, u) \leq u$. Suppose now that p, q are such that $(\lambda^* y(p)), (\lambda^* y(p))$ when applied to u satisfy the inequalities (5.18.16). Then, $(\lambda^* y(pq))u = \mathrm{s}(\lambda^* y(p))(\lambda^* y(q))u \leq (\lambda^* y(p))u((\lambda^* y(q))u) \leq p\{y := u\}q\{y := u\} = pq\{y := u\}$.

Observe that, since $\langle X \rangle$ contains k, s and is closed under applications, then the condition on the coefficients of $\lambda^* y(t)$ follows by induction. □

Observation 5.19. (1) Sometimes we write $\lambda^* y(t) = \lambda^* y.t$

(2) Since application is monotone in both arguments, the proof of Theorem 5.18 can be interpreted as a method to translate lambda terms into elements of $A[\mathcal{V}]$ in such a way that $\leq$ reflects β-reduction.
(3) Moreover, the condition on the coefficients guarantees that lambda terms are translated as polinomials with coefficients on $\langle \emptyset \rangle$ which is included into any filter Φ (it is in fact the minimal filter of A). In particular, a closed lambda term is translated as a constant polynomial with coefficients on Φ, which is identified with an element of Φ.

Theorem 5.20. *If A is an $\mathcal{OCA}$, then:*

(1) *There are elements $p, p_1, p_2 \in \Phi$ with the following properties:*

$$\forall a, b \in A\ ,\ p_1(pab) \leq a\ ;\ p_2(pab) \leq b. \tag{5.20.17}$$

It is customary to call $pab = a \wedge b$ *and in that case the properties above –Equation* (5.20.17)*– read:*

$$\forall a, b \in A\,,\; p_1(a \wedge b) \leq a\,;\; p_2(a \wedge b) \leq b. \tag{5.20.18}$$

(2) *There is an* $f \in \Phi$ *such that for all* $a, b \in A$ *we have that*

$$(fa)b \leq ba. \tag{5.20.19}$$

(3) *There are functions* $D, E, F, G : A \to A$ *and* $M : A \times A \to A$ *such that for all* $a, b, c \in A$*, then:*

$$((D(a)c)b) \leq c(ab) \quad , \quad ((E(a)b)c) \leq c(ab) \tag{5.20.20}$$

$$(F(c)a)b \leq c(ab) \tag{5.20.21}$$

$$G(c)(pab) \leq (ca)b \tag{5.20.22}$$

$$M(c, b)a \leq (ca)b. \tag{5.20.23}$$

Moreover: $D(\Phi) \subseteq \Phi, E(\Phi) \subseteq \Phi\,,\; F(\Phi) \subseteq \Phi\,,\; G(\Phi) \subseteq \Phi$ *and* $M(\Phi, \Phi) \subseteq \Phi$.

Proof. (1) Define $p = \lambda^* x_1 \lambda^* x_2 \lambda^* x_3 x_3 x_1 x_2$, $p_1 = \lambda^* x_1 x_1\, \mathrm{k}$, $p_2 = \lambda^* x_1 x_1\, \mathrm{k}'$; where $\mathrm{k}' = \lambda^* x_1.\lambda^* x_2.x_2$.

(2) Consider $f = \lambda^* x_1 \lambda^* x_2 x_2 x_1$. In this situation it is clear that $(fa)b \leq ba$.

(3) Define $D(a) = \lambda^* x \lambda^* y x(ay)$, $E(a) = \lambda^* x \lambda^* y y(ax)$, $F(c) = \lambda^* x \lambda^* y c(xy)$, $G(c) = \lambda^* x (c(p_1 x))(p_2 x)$ and $M(c, b) = \lambda^* x.(cx)b$.

□

For later use we prove some properties of the *meet* or *wedge* operator.

Lemma 5.21. *Assume that A is an* $\mathcal{OCA}$ *as above –Definition 5.9–.*

(1) *The operator* $\wedge : A \times A \to A$ *is monotone in both variables, i.e.* $a \leq a', b \leq b'$ *implies that* $a \wedge b \leq a' \wedge b'$

(2) *There is a map* $R : A \to A$ *with the property that for all* $a, b, c \in A$ *we have that* $R(c)(a \wedge b) \leq a \wedge (cb)$. *Moreover* $R(\Phi) \subseteq \Phi$.

Proof. (1) This part follows directly from the fact that the application in A is monotone in both variables.

(2) The following chain of inequalities yields the result.

$$a \wedge (cb) = (pa)(cb) \geq (D(c)(pa))b \geq \big((F(D(c))p)a\big)b \geq G(F(D(c))p)(pab) \geq R(c)(a \wedge b).$$

Where we denoted $G(F(D(c))p) = R(c)$. The justification of the chain of inequalities is the following going from left to right: (5.20.20), (5.20.21), (5.20.22).

□

We need some consequences of Theorem 5.20, that we record here for later use.

Corollary 5.22. (1) *There is a function* $H : A \times A \to A$ *with the property that for all* $a, b, c, m, n \in A$*, we have that:*

$$m((na)b) \leq c \Rightarrow H(m, n)a \leq (b \to c). \tag{5.22.24}$$

Moreover, the function H satisfies that $H(\Phi, \Phi) \subseteq \Phi$.

(2) *In the previous notations, for any* $a, b, c \in A$ *we have that*

$$(F(\mathrm{e})F(c))(a \to b) \leq (a \to (cb)).$$

(3) *In the previous notations, for any* $a, b \in A$ *we have that*

$$(F(\mathrm{e})f)a \leq b \to ba.$$

In particular

$$(F(\mathrm{e})f)a \leq \mathrm{i} \to a.$$

(4) *For all* $a, b \in A$ *as for* $f \in \Phi$ *as above, we have that:*

$$(fb)(b \to a) \leq a.$$

In particular

$$(f\,\mathrm{i})(\mathrm{i} \to a) \leq a.$$

Proof. The proof follows from previous constructions.

(1) By applying a few times inequality (5.20.21) we have that:

$$((F^2(m)n)a)b \leq (F(m)(na))b \leq m((na)b) \leq c.$$

By the basic property of the adjunctor we deduce that: $\mathrm{e}((F^2(m)n)a) \leq (b \to c)$. Using again the inequality (5.20.21) we obtain that: $\big(F(\mathrm{e})(F^2(m)n)\big)a \leq \mathrm{e}\,((F^2(m)n)a) \leq (b \to c)$. Then, this part is proved by taking $H(m,n) = F(\mathrm{e})(F^2(m)n)$.

(2) Starting from $(a \to b) \leq (a \to b)$ we deduce that $(a \to b)a \leq b$ and then $c((a \to b)a) \leq cb$. By using inequality (5.20.21) we deduce that $(F(c)(a \to b))a \leq cb$ and by the property of the adjunctor we deduce that $e(F(c)(a \to b)) \leq (a \to cb)$. Then, the proof can be finished using again inequality (5.20.21).

(3) Starting from –see (5.20.19)– $(fa)b \leq ba$ we deduce that $\mathrm{e}(fa) \leq b \to ba$. Using (5.20.21), we conclude that $(F(\mathrm{e})f)a \leq b \to ba$. The rest of the assertion is guaranteed if we take $b = \mathrm{i}$.

(4) By the definition of f we have that $fb(b \to a) \leq (b \to a)b$ which is less or equal than a.

□

Observation 5.23. (1) Concerning the converse of the above Corollary 5.22, one has the following easy result that is a direct consequence of the inequality (5.20.22) defining the function G.
For the function $G : A \to A$ we have that for all $m, a, b, c \in A$:

$$ma \leq (b \to c) \Rightarrow G(m)(a \wedge b) = G(m)((pa)b) \leq c.$$

Indeed, the basic half adjunction property guarantees that $ma \leq (b \to c) \Rightarrow (ma)b \leq c$. The rest follows from the definition of G. Moreover, the function G is such that $G(\Phi) \subset \Phi$ and the element $p \in \Phi$.

(2) It is interesting to consider the following. The proof of Corollary 5.22 uses strongly the property of the existence of the adjunctor e in the $\mathcal{OCA}$. Here we show a converse, i.e. if the result (5.22.24) is valid, the existence of the adjunctor can be deduced.
Indeed, if we assume that $ab \leq c$, applying twice the fact that $\mathrm{i}\,d \leq d$ for all $d \in A$, we conclude that $\mathrm{i}((\mathrm{i}\,a)b) \leq (\mathrm{i}\,a)b \leq ab \leq c$. Hence using the result of Corollary 5.22, we deduce that $H(\mathrm{i},\mathrm{i})a \leq (b \to c)$. Hence, the element $\mathrm{e} = H(\mathrm{i},\mathrm{i}) \in \Phi$, works as an adjunctor.

26. In what follows we construct in an $\mathcal{OCA}$ with a filter Φ a new partial order (not necessarily antisymmetric) that will be used to construct a tripos from the $\mathcal{OCA}$.

Definition 5.24. Assume that the quintuple $(A, \circ, \leq, \mathrm{k}, \mathrm{s}, \Phi)$ is an $\mathcal{OCA}$ with a filter. We define the relation $\sqsubseteq_\Phi$ in A as follows:

$$a \sqsubseteq_\Phi b, \text{if and only if } \exists f \in \Phi : f \circ a \leq b.$$

Usually we omit the subscript Φ in the notation of the relation $\sqsubseteq_\Phi$, and as usual omit the symbol $\circ$ when dealing with the application in A that is written $a \circ b = ab$.

Lemma 5.25. *In the context of Definition 5.24, we have the following properties of* $\sqsubseteq$.

(1) *The relation* $\sqsubseteq$ *is a partial order in* A *–not necessarilty antisymmetric–.*

(2) *The partial order* $\leq$ *is stronger than* $\sqsubseteq$ *(i.e. if* $a \leq b$*, then* $a \sqsubseteq b$*).*

(3) *The order* $\sqsubseteq$ *has the following compatibility relation with the application on* A*: for all* $a, a', b, b' \in A$ *we have that*

$$a \sqsubseteq b \quad \text{and} \quad a' \sqsubseteq b' \Rightarrow a \wedge a' \sqsubseteq bb'.$$

(4) *If* $f \sqsubseteq (a \to b)$ *with* $f \in \Phi$*, then* $a \sqsubseteq b$*.*

(5) *If* A *has an adjunctor, then for all* $a, b \in A$*,* $a \sqsubseteq b$ *if and only if there is an element* $f \in \Phi$ *such that* $f \leq a \to b$*.*

Proof. (1) (a) $a \sqsubseteq a$ is a consequence of the fact that $\mathrm{i}\,a \leq a$ –see Lemma 5.13. Observe that being Φ closed under the operation of A, the element $\mathrm{i} \in \Phi$.

(b) If $a \sqsubseteq b$ and $b \sqsubseteq c$, then $a \sqsubseteq c$. Indeed, by definition we can find $g, f \in \Phi$ such that:

$$ga \leq b \quad , \quad fb \leq c,$$

and using the monotony of the operation of A we deduce that $f(ga) \leq fb \leq c$. Using Theorem 5.20,(3) we deduce that there is an $h \in \Phi$ such that $ha \leq f(ga) \leq c$, that is our conclusion.

(2) Suppose that $a \leq b$, then $\mathrm{i}\, a \leq a \leq b$ so that $a \sqsubseteq b$.

(3) By hypotesis, there exist $f, f' \in \Phi$ with the property that: $fa \leq b$ and $f'a' \leq b'$. Call $a_0 = a \wedge a'$ and recall that $p_1 a_0 \leq a$ and $p_2 a_0 \leq a'$ as in Theorem 5.20 (1). Then $(F(f)p_1)a_0 \leq f(p_1 a_0) \leq fa$ and $(F(f')p_2)a_0 \leq f'(p_2 a_0) \leq f'a'$ –see (5.20.21). If we abbreviate: $g_1 = F(f)p_1, g_2 = F(f)p_2$ we deduce that $(g_1 a_0)(g_2 a_0) \leq bb'$. Using the basic property of s we obtain that $\mathrm{s}\, g_1 g_2 a_0 \leq (g_1 a_0)(g_2 a_0) \leq bb'$ and reducing again using the inequality (5.20.21) we deduce that for some $h \in \Phi$ –depending only on s, g_1, g_2–, it is verified that $ha_0 \leq bb'$. This is our conclusion.

(4) If $a \sqsubseteq b$, then for some $f \in \Phi$ we have that $fa \leq b$, then $ef \leq a \to b$. Conversely, if $f \leq a \to b$ for $f \in \Phi$, then $fa \leq b$ and $a \sqsubseteq b$.

(5) If $f \sqsubseteq (a \to b)$, then there is a $g \in \Phi$ such that $gf \leq (a \to b)$ and then $(gf)a \leq b$ and then $a \sqsubseteq b$.

□

The theorem that follows, guarantees the complete adjunction property in an $\mathcal{OCA}$ with adjunctor, with respect to the order $\sqsubseteq$, the "meet" operation and the arrow. It will be important for the categorification of the structures.

Theorem 5.26. *If the original $\mathcal{OCA}$ has an adjunctor, then the partial order $\sqsubseteq$ satisfies the following* "adjunction property" *with respect to the operations $\wedge, \to$*[6]:

$$a \wedge b \sqsubseteq c \Leftrightarrow a \sqsubseteq (b \to c).$$

Proof. Assume that $a \sqsubseteq (b \to c)$, then for some $f \in \Phi$, $fa \leq (b \to c)$ and then $(fa)b \leq c$. From the inequality (5.20.22), we deduce that $G(f)(pab) \leq (fa)b \leq c$ and then that $a \wedge b \sqsubseteq c$.

Conversely, if we assume that $a \wedge b \sqsubseteq c$, then $f((pa)b) \leq c$ for some $f \in \Phi$. Then applying the inequality (5.20.21), we deduce that $(F(f)(pa))b \leq f((pa)b) \leq c$. Applying again the same inequality to the first factor we obtain that $((F^2(f)p)a)b \leq (F(f)(pa))b \leq c$ and then we deduce that: $\mathrm{e}((F^2(f)p)a) \leq (b \to c)$. Using once again the inequality (5.20.21) we obtain that $(F(\mathrm{e})(F^2(f)p))a \leq \mathrm{e}((F^2(f)p)a) \leq (b \to c)$, then as $\mathrm{e}, F(\mathrm{e}), f, F(f), F^2(f), p \in \Phi$, we conclude that $a \sqsubseteq (b \to c)$. □

6. Construction of an $\mathcal{OCA}$ from a $\mathcal{PAKS}$.

27. In this section we show how to perform a natural construction of an $\mathcal{OCA}$ from a $\mathcal{PAKS}$.

Definition 6.1. Assume we have a $\mathcal{PAKS}$

$$(\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{app}, \mathrm{save}, \mathrm{push}, \mathrm{K}, \mathrm{S}, \mathrm{cc}),$$

and define a set A, an order, an application, an implication, the combinators k, s, and an adjunctor e in the following manner:

(1) $A = \mathcal{P}_\perp(\Pi)$;

(2) For a pair of elements $a, b \in A$ we say that $a \leq b$ iff $a \supseteq b$.

(3) For a pair of elements $a, b \in A$ we define $a \circ b$ as in Definition 3.4, (1). In other words:

$$a \circ b = ({}^{\perp}\{\pi \in \Pi : \forall t \in {}^{\perp}a, \forall s \in {}^{\perp}b \quad t \perp s.\pi\})^{\perp} = ({}^{\perp}\{\pi \in \Pi : {}^{\perp}a \subseteq ({}^{\perp}b.\pi)^{\perp}\})^{\perp} = ({}^{\perp}\{\pi \in \Pi : a \supseteq ({}^{\perp}b.\pi)\})^{\perp}.$$

(4) For a pair of elements $a, b \in A$ we define $a \to b$ as in Definition 3.4, (2). In other words:

$$a \to b = ({}^{\perp}\,\mathrm{push}({}^{\perp}a, b))^{\perp} = ({}^{\perp}(a^{\perp} \cdot b))^{\perp}.$$

(5) We define the following elements of A:

$$\mathrm{k} = \{\pi \in \Pi : \mathrm{K} \perp \pi\} = \{\mathrm{K}\}^{\perp}\,,\ \mathrm{s} = \{\pi \in \Pi : \mathrm{S} \perp \pi\} = \{\mathrm{S}\}^{\perp}.$$

[6] Reading the proof the reader may verify that the existence of the adjunctor is not necessary to prove the assertion: $a \sqsubseteq (b \to c) \Rightarrow a \wedge b \sqsubseteq c$.

(6) We define $e = \{E\,E\}^{\perp}$.

28. We prove the following crucial theorem.

Theorem 6.2. *Consider the* $\mathcal{PAKS}$*:*

$$(\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{app}, \mathrm{save}, \mathrm{push}, \mathrm{K}, \mathrm{S}, \mathrm{cc}),$$

and the quintuple as presented in Definition 6.1:

$$(A, \leq, \circ, \mathrm{k}, \mathrm{s}).$$

The above quintuple forms an $\mathcal{OCA}$*. Moreover, the map* $\to$ *is an implication and the element* e *is an adjunctor and if the* $\mathcal{AKS}$ *is classical, so is the* $\mathcal{OCA}$*.*

Proof. It is clear that $\circ$ is an application in A, that $\leq$ is a partial order, and we have defined the elements k and s. Concerning the monotony of the application we have to prove that if: $a \supseteq a'$, $b \supseteq b'$, then if $\pi \in \Pi$ satisfies that $a' \supseteq ({}^{\perp}b'.\pi)$, then $a \supseteq ({}^{\perp}b.\pi)$.

We have that $a \supseteq a' \supseteq ({}^{\perp}b'.\pi)$. As $b \supseteq b'$, ${}^{\perp}b' \supseteq {}^{\perp}b$ and then $({}^{\perp}b'.\pi) \supseteq ({}^{\perp}b.\pi)$ and the proof of the monotony of $\circ$ is finished.

The monotony and antimonotony of the map $\to$ is similarly proved. The fact that the arrow $\to$ satisfies the *half adjunction property*: if $a \leq (b \to c)$ then $ab \leq c$, was established in Theorem 3.6.

Next, we prove that $\mathrm{k}\,ab \leq a$. We have seen that $\mathrm{K} \in ({}^{\perp}a.({}^{\perp}b.a))^{\perp}$ and that means that $\{\mathrm{K}\} \subseteq ({}^{\perp}a.({}^{\perp}b.a))^{\perp}$ that implies $\mathrm{k} \supseteq \left({}^{\perp}({}^{\perp}a.({}^{\perp}b.a))\right)^{\perp} \supseteq {}^{\perp}a.({}^{\perp}b.a)$.

Now, from the above inclusion we deduce that: $\mathrm{k}\,a = ({}^{\perp}\{\pi \in \Pi : \mathrm{k} \supseteq {}^{\perp}a.\pi\})^{\perp} \supseteq \{\pi \in \Pi : \mathrm{k} \supseteq {}^{\perp}a.\pi\} \supseteq {}^{\perp}b.a$, or in other words that $\mathrm{k}\,a \leq (b \to a)$ –see Definition 3.4, (2)–. Using the half adjunction property 3.6, we deduce that $\mathrm{k}\,ab \leq a$.

The condition $\mathrm{s}\,abc \leq (ac)(bc)$ can be proved similarly.

Indeed, it is enough to prove that $\mathrm{s}\,ab \leq c \to (ac)(bc)$ that means that $({}^{\perp}\{\pi : \mathrm{s}\,a \supseteq {}^{\perp}b.\pi\})^{\perp} \supseteq \left({}^{\perp}({}^{\perp}c.(ac)(bc))\right)^{\perp}$. Then, it is enough to prove that $\{\pi : \mathrm{s}\,a \supseteq {}^{\perp}b.\pi\} \supseteq {}^{\perp}c.(ac)(bc)$ or $\mathrm{s}\,a \supseteq {}^{\perp}b.{}^{\perp}c.(ac)(bc)$. Now, as $\mathrm{s}\,a = \{\pi \in \Pi : \mathrm{s} \supseteq {}^{\perp}a.\pi\}$ we have to check that $\mathrm{s} \supseteq {}^{\perp}a.{}^{\perp}b.{}^{\perp}c.(ac)(bc)$ or equivalently that $\mathrm{S} \perp {}^{\perp}a.{}^{\perp}b.{}^{\perp}c.(ac)(bc)$ or $\mathrm{S} \in {}^{\perp}({}^{\perp}a.{}^{\perp}b.{}^{\perp}c.(ac)(bc))$.

Hence, we take $t \perp a$, $s \perp b$, $u \perp c$ and $\pi \in (ac)(bc)$ and using Lemma 4.5 (S1) (1), we deduce that $tu(su) \perp \pi$.

Using now **11.** condition (S3), we prove that $\mathrm{S} \perp t.s.u.\pi$, that is the result we want.

Finally the proof that e as introduced above –Definition 6.1–, is an adjunctor is the content of Theorem 4.12.

If we take $\mathrm{c} = \mathrm{cc}^{\perp}$, we proved in Lemma 4.5, ($\mathbb{S}$4) that $\mathrm{cc} \in {}^{\perp}(((a \to b) \to a) \to a)$, that implies that $\mathrm{c} \supseteq ({}^{\perp}(((a \to b) \to a) \to a))^{\perp} = (((a \to b) \to a) \to a)$, i.e. $\mathrm{c} \leq (((a \to b) \to a) \to a)$. □

7. Construction of an $\mathcal{OCA}$ with a filter from an $\mathcal{AKS}$.

29. Assume that we have a decuple

$$(\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{app}, \mathrm{save}, \mathrm{push}, \mathrm{K}, \mathrm{S}, \mathrm{cc}, \mathrm{QP}),$$

where the first nine elements define a $\mathcal{PAKS}$ and the last $\mathrm{QP} \subseteq \Lambda$ is a subset of terms that contains the distinguished elements K, S and cc and is closed by application.

Definition 7.1. Define the subset Φ of $A = \mathcal{P}_{\perp}(\Pi)$ as follows:

$$\Phi = \{f \in A : {}^{\perp}f \cap \mathrm{QP} \neq \emptyset\} = \{f \in A : \exists t \in \mathrm{QP}, t \perp f\}.$$

Lemma 7.2. *The subset* $\Phi \subseteq A$ *is a filter in* A *–see Definition* 5.14 *–that contains* e *and* c*.*

Proof. (F1) If $f \in \Phi$ and $a \in A$ is $f \leq a$, then $a \in \Phi$. This is because, by hypothesis we have an element $t_f \in {}^{\perp}f \cap \mathrm{QP} \subseteq {}^{\perp}a \cap \mathrm{QP}$. Hence, $a \in \Phi$.

(F2) The subset Φ is closed under application because in accordance with Lemma 4.5, (S1), (5), if $t_f \in {}^{\perp}f \cap \mathrm{QP}$ and $t_g \in {}^{\perp}g \cap \mathrm{QP}$ then $t_f t_g \in {}^{\perp}f^{\perp}g \cap \mathrm{QP} \subseteq {}^{\perp}(f \circ g) \cap \mathrm{QP}$.
(F3) $\mathrm{k}, \mathrm{s} \in \Phi$ because $\mathrm{K} \in {}^{\perp}\mathrm{k} \cap \mathrm{QP}$ and $\mathrm{S} \in {}^{\perp}\mathrm{s} \cap \mathrm{QP}$.
(F4) $\mathrm{e} \in \Phi$ because $\mathrm{EE} \in {}^{\perp}\mathrm{e} \cap \mathrm{QP}$ –see Observation 4.14.
(F5) Being $\mathrm{c} = \{\mathrm{cc}\}^{\perp}$, it is clear that: $\mathrm{cc} \in {}^{\perp}\mathrm{c} \cap \mathrm{QP}$.

□

30. Now we have enough machinery in order to answer the following question: *Is the filter built as above closed under meets?*

Assume that it is closed under meets. In this case there is an element $\Omega \in \Phi$ with the property that $\Omega \leq \mathrm{k}$ and also $\Omega \leq \mathrm{s}$.

(1) In that situation a direct computation guarantees that $\Omega\Omega\,\mathrm{k}\,\mathrm{k}$ is at the same time $\Omega\Omega\,\mathrm{k}\,\mathrm{k} \leq k$ and $\Omega\Omega\,\mathrm{k}\,\mathrm{k} \leq \mathrm{sk}$. Indeed, $\Omega\Omega\,\mathrm{k}\,\mathrm{k} \leq \mathrm{skkk} \leq \mathrm{kk(kk)} \leq \mathrm{k}$. Also, $\Omega\Omega\,\mathrm{k}\,\mathrm{k} \leq \mathrm{kskk} \leq \mathrm{sk}$.
(2) Hence, given $f, g \in \Phi$ we have that $\Omega\Omega\,\mathrm{k}\,\mathrm{k}\, fg \leq \mathrm{k}\, fg \leq f$ and also: $\Omega\Omega\,\mathrm{k}\,\mathrm{k}\, fg \leq \mathrm{sk}\, fg \leq \mathrm{k}\, g(fg) \leq g$. Then, in this situation for any pair $f, g \in \Phi$ the element $\Omega\Omega\,\mathrm{k}\,\mathrm{k}\, fg \leq f$ and also $\Omega\Omega\,\mathrm{k}\,\mathrm{k}\, fg \leq g$.
(3) Consider an AKS and the corresponding $\mathcal{OCA}$. We have that $\Omega \supseteq \mathrm{s} \cup \mathrm{k}$ and ${}^{\perp}\Omega \subseteq {}^{\perp}(\{\mathrm{S}\}^{\perp} \cup \{\mathrm{K}\}^{\perp}) \subseteq {}^{\perp}(\{\mathrm{S}\}^{\perp}) \cap {}^{\perp}(\{\mathrm{K}\}^{\perp})$.
(4) The above condition means that: $\forall Q \in {}^{\perp}\Omega, \forall \pi \in \Pi, (S \star \pi \in \perp\!\!\!\perp \Rightarrow Q \star \pi \in \perp\!\!\!\perp)$ and $(\mathrm{K} \star \pi \in \perp\!\!\!\perp \Rightarrow Q \star \pi) \in \perp\!\!\!\perp$.
(5) Consider the $\mathcal{AKS}$ defined by the $\mathcal{KAM}$ with only substitutive and deterministic instructions, defining $\perp\!\!\!\perp = \{t \star \pi \succ \mathrm{S} \star \alpha$ or $t \star \pi \succ \mathrm{K} \star \beta\}$, with α, β different stack constants. Since, $\alpha \in \{\mathrm{S}\}^{\perp}, \beta \in \{\mathrm{K}\}^{\perp}$ we get $\alpha, \beta \in {}^{\perp}\Omega$. In this situation: if $Q \in {}^{\perp}\Omega$ we get from the statement above $Q \star \alpha, Q \star \beta \in \perp\!\!\!\perp$, since $\alpha \in \{\mathrm{K}\}^{\perp}, \beta \in \{\mathrm{S}\}^{\perp}$. By definition of $\perp\!\!\!\perp$: $Q \star \alpha \succ \mathrm{S} \star \alpha$ or $Q \star \alpha \succ \mathrm{K} \star \beta$.
(6) Assume now that $Q \in \mathrm{QP}$. Then q does not contain k_π and cannot change the stack constant, and hence: $Q \star \alpha \succ \mathrm{S} \star \alpha$. By substitution $Q \star \beta \succ \mathrm{S} \star \beta$. But, again because Q cannot change the stack constant and $q \star \beta \in \perp\!\!\!\perp$, we get $Q \star \beta \succ \mathrm{K} \star \beta$. Thus we obtain $\mathrm{S} \star \beta \succ \mathrm{K} \star \beta$ or $\mathrm{K} \star \beta \succ \mathrm{S} \star \beta$ which is impossible because both $\mathrm{K} \star \beta$ and $\mathrm{S} \star \beta$ does not recduce because they does not have arguments. Then, we conclude that $\Omega^{\perp} \cap \mathrm{QP} = \emptyset$. This contradicts the assumption that $\Omega \in \Phi$.

A model where it is true that a pair of elements of Φ always has a minimum is when $\perp\!\!\!\perp = \emptyset$. Here $\mathrm{s} = \{\mathrm{S}\}^{\perp} = \{\mathrm{K}\}^{\perp} = \mathrm{k}$, being $\mathrm{s} = \mathrm{k}$ the set Φ is a filter in the usual sense.

8. From $\mathcal{OCA}$s to Tripos

31. Assume we have an $\mathcal{OCA}$: $(A, \circ, \leq, \mathrm{k}, \mathrm{s})$, that is equipped with an implication, an adjunctor and a filter –called respectively: $\rightarrow$, e and Φ.

Let I be an arbitrary set and consider A^I the cartesian product of I copies of A –viewed in general as the set of functions $A^I = \{\varphi : I \to A : \varphi \text{ is a function}\}$.

Observation 8.1. We consider some properties of the order and the operations in an $\mathcal{OCA}$ and its extensions to cartesian products.

(1) We have the following orders in A^I.
  (a) *Cartesian product of* $\leq$: If $\varphi, \psi \in A^I$, $\varphi \leq \psi$ if and only if $\forall i \in I : \varphi(i) \leq \psi(i)$.
  (b) *Cartesian product of* $\sqsubseteq$: If $\varphi, \psi \in A^I$, $\varphi \sqsubseteq \psi$ if and only if $\forall i \in I, \exists f_i \in \Phi : f_i\varphi(i) \leq \psi(i)$.
  (c) *Entilement order*: If $\varphi, \psi \in A^I$, $\varphi \vdash \psi$ if and only if $\exists f \in \Phi, \forall i \in I : f\varphi(i) \leq \psi(i)$.
(2) In the case that Φ has inf, it is clear that the orders listed in (b) and (c) above, are equivalent.
(3) Clearly the first order above is reflexive, antisymmetric and transitive; the second and third orders are reflexive and transitive. The proof of these last properties are identical to the proofs of the corresponding properties of the order $\sqsubseteq$ in A.
(4) One can define the arrow in A^I simply as: $(\varphi \to \psi)(i) = \varphi(i) \to \psi(i)$.
(5) The "meet" in A^I can be defined as $(\varphi \wedge \psi)(i) = \varphi(i) \wedge \psi(i) = p\varphi(i)\psi(i)$.
(6) A manner to view the entilement order is the following. Assume that we take A to be an $\mathcal{OCA}$ as above and that M is an A–module, i,e, a set M together with an operation $(a, m) \mapsto a.m : A \times M \to M$. The

standard example of an A–module is A^I, with the operation $(a.\psi)(i) = a\psi(i)$. If we have a partial orden $\leq_M \subset M \times M$, we can define a new order $\sqsubseteq_M \subset M \times M$ as follows: if $m, n \in M$ we say that $m \sqsubseteq_M n$ if and only if there exist an element $f \in \Phi$ such that $f.m \leq_M n$. In this sense the order appearing in (1)(c) above –the entilement order– is obtained from the cartesian product order appearing in (1)(a), by the process just mentioned.

32. The following "complete adjunction property –or simply adjunction property–" of the order "entile" is important. It is worth noticing that it does not follow directly from the corresponding property proved for A in Theorem 5.26 –i.e the property valid for all $a, b, c \in A$ thay states that $a \wedge b \sqsubseteq c \Leftrightarrow a \sqsubseteq (b \to c)$–. We need the subtler properties given in Corollary 5.22 and Observation 5.23.

Theorem 8.2. *In the notations above for an $\mathcal{OCA}$ with implication, adjunctor and filter, the following is true for all $\varphi, \psi, \theta \in A^I$:*

$$\varphi \wedge \psi \vdash \theta \Longleftrightarrow \varphi \vdash (\psi \to \theta).$$

Proof. $\Longrightarrow$ Take $f \in \Phi$ such that $f(p\varphi(i)\psi(i)) \leq \theta(i)$ for all $i \in I$. Using Corollary 5.22 we obtain that $H(f, p)\varphi(i) \leq (\psi(i) \to \theta(i))$. Hence, we deduce that $\varphi \vdash (\psi \to \theta)$.

$\Longleftarrow$ Assume that for $f \in \Phi$ we have that $f\varphi(i) \leq (\psi(i) \to \theta(i))$ for all $i \in I$. Then, in accordance with Observation 5.23, (1) we deduce that $G(f)((p\varphi(i))\psi(i)) = G(f)(\varphi(i) \wedge \psi(i)) \leq \theta(i)$ for all $i \in I$. In other words we have proved that: $\varphi \wedge \psi \vdash \theta$.

□

33. Next we add some structure in order to continue with the construction of the tripos. For an arbitrary subset $X \subset A$ of the $\mathcal{OCA}$, there is an element $\inf(X) \in A$ that is the infimum of X with respect to the order $\leq$.

Definition 8.3. Let us consider that we have an $\mathcal{OCA}$ $(A, \leq, \circ, \mathrm{s}, \mathrm{k})$, equipped with an implication $\to$, an adjunctor e and a filter Φ as seen in Definition 5.9. This $\mathcal{OCA}$ is said to be a ${}^{\mathcal{K}}\mathcal{OCA}$ if it is inf–*complete*; i.e.: if the operator $\inf : \mathcal{P}(A) \to A$ is everywhere defined.

Definition 8.4. We define the element $\bot \in A$ as $\bot = \inf A$.

We list a few basic properties of the operations in the $\mathcal{OCA}$ in relation with the element $\bot$.

Lemma 8.5. *Let us assume that A is a ${}^{\mathcal{K}}\mathcal{OCA}$ (c.f. Definition 8.3), then:*

(1) *For all $a \in A$ we have that $\bot\ a = \bot$.*
(2) *If $b \leq (\mathrm{i} \to a)$, then $\mathrm{e}(\mathrm{si}\, b) \leq (\mathrm{i} \to a)$ for $a, b \in A$. In particular in the usual notation for the function F –see Theorem 5.20– we have that $(F(\mathrm{e})(\mathrm{si}))(\mathrm{i} \to a) \leq (\mathrm{i} \to a)$.*
(3) *If $a \in A$, then $\mathrm{si}\, a\ \bot = \bot$. Moreover, for all $a, b \in A$ we have that $(F(\mathrm{e})(\mathrm{si}))a \leq (\bot \to b)$.*

Proof. (1) Clearly as $\bot \leq (a \to \bot)$ we deduce that $\bot\ a \leq \bot$ then $\bot\ a = \bot$.

(2) We have that $\mathrm{si}\, b\, \mathrm{i} \leq \mathrm{ii}(b\,\mathrm{i}) \leq \mathrm{i}(b\,\mathrm{i}) \leq b\,\mathrm{i} \leq a$, the last equality coming from the hypothesis that $b \leq (\mathrm{i} \to a)$. Hence, from the basic property of the adjuntor we obtain that: $\mathrm{e}(\mathrm{si}\, b) \leq (\mathrm{i} \to a)$. If we apply the above result to the case that $b = (\mathrm{i} \to a)$, and then Theorem 5.20, we obtain the second part of the conclusion.

(3) We have that: $\mathrm{si}\, a\ \bot \leq \mathrm{i}\ \bot\ (a\ \bot) \leq \bot\ (a\ \bot) = \bot$ where the first inequality comes from the characterization of s the second from the characterization of i and the third was proved in (1). Hence, $\mathrm{si}\, a\ \bot \leq b$ for all b. From the basic property of the adjunctor we deduce that $\mathrm{e}(\mathrm{si}\, a) \leq (\bot \to b)$ and the proof is finished proceeding in the same way than in part (2).

□

Observation 8.6. (1) Notice that we have in particular proved the following assertion that follows directly from parts (2) and (3) of the above Lemma 8.5: there is an element $g \in \Phi$ such that for all $a, b \in A$: $g(\mathrm{i} \to a) \leq (\mathrm{i} \to a)$ and $g(\mathrm{i} \to a) \leq (\bot \to b)$.

(2) In fact the inequality in part (3) guarantees that for all a, b:

$$ga \leq (\bot \to b).$$

Definition 8.7. Given I a set we define the equality predicate in $A^{I\times I}$ as follows: $\mathrm{eq}_I : I \times I \to A$

$$\mathrm{eq}_I(i,j) = \begin{cases} \mathrm{i} = \mathrm{skk} & \text{if} \quad i = j; \\ \perp & \text{if} \quad i \neq j. \end{cases}$$

It is clear that for all $a, b \in A$ and for all $i, j \in I$:
(1) $\mathrm{eq}_I(i,i)a \le a$,
(2) $\mathrm{eq}_I(i,j)a \le b$ if $i \neq j$.

34. More can be said about $\mathcal{OCA}$s coming from $\mathcal{AKS}$s.

Observation 8.8. As we have seen in Sections 6 and 7, given an $\mathcal{AKS}$ we can produce an $\mathcal{OCA}$ that is simply $A = \mathcal{P}_\perp(\Pi)$ with the order $\le$ given by the reverse inclusion and with a filter Φ defined as the set of elements of A that are realized by some element of the set of quasi proofs $\mathrm{QP} \subseteq \Lambda$. The rest of the ingredients $\circ, \to, \mathrm{s}, \mathrm{k}, \mathrm{e}$ are defined as before –see in particular Theorem 6.2 and Lemma 7.2.

Notice that for this particular kind of $\mathcal{OCA}$s, *both* sup and inf can be defined. Indeed if $X \subset \mathcal{P}_\perp(\Pi) = A$, then $\inf(X) = ({}^\perp(\bigcup X))^\perp$ and $\sup(X) = ({}^\perp(\bigcap X))^\perp$.

In particular $\mathcal{P}_\perp(\Pi)$ is an inf–complete $\mathcal{OCA}$.

35. Let A be an $\mathcal{OCA}$ and we will work in the category denoted as $[\mathrm{Set}^{\mathrm{op}}, \mathrm{Preorder}] = \mathrm{Preorder}^{\mathrm{Set}^{\mathrm{op}}}$, that has as objects the functors $F : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Preord}$, and as arrows the natural transformations between functors. The category Preord is the category whose objects are the partially order sets and its arrows are the monotone functions between the partially ordered sets.

Definition 8.9. Given the $\mathcal{OCA}$ called A we define the "regular functor" $\mathcal{R}_A \in \mathrm{Preord}^{\mathrm{Set}^{\mathrm{op}}}$ as follows:

$$\mathcal{R}_A : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Preord},$$

with $\mathcal{R}_A(I) = (A^I, \vdash)$ where $\vdash$ is as in the definition appearing in Observation 8.1 item **??**.

If $\alpha : J \to I$, then $\alpha^* = \mathcal{R}_A(\alpha) : (A^I, \vdash) \to (A^J, \vdash)$ is defined as: $\alpha^*(\varphi) = \varphi \circ \alpha$.

Observation 8.10. (1) To prove that the above Definition 8.9 makes sense, we have to check that α^* is monotone in relation with the order of entilement: if $\varphi, \varphi' \in A^I$, and $\varphi \vdash \varphi'$, then $\alpha^*(\varphi) \vdash \alpha^*(\varphi')$. We have to prove that if there is an $f \in \Phi$ with the property that $f\varphi(i) \le \varphi'(i)$ for all $i \in I$, then there is a $g \in \Phi$ such that for all $j \in J$: $g\varphi(\alpha(j)) = \varphi'(\alpha(j))$. This is clearly true by taking $f = g$.
(2) It is clear that $\mathcal{R}_A(\alpha\beta) = \mathcal{R}_A(\beta)\mathcal{R}_A(\alpha)$.

Next we define another functor, with the same object part than $\mathcal{R}_A$. That will be the "right adjoint" of $\mathcal{R}_A$.

Definition 8.11. Define the functor $\forall_A : \mathrm{Set} \to \mathrm{Preord}$. At the level of objects $\forall_A(I) = (A^I, \vdash)$, and for an arrow $\alpha : J \to I$ and $\varphi : J \to A$, we define $\forall_A\alpha(\varphi) : I \to A$ as $\forall_A\alpha(\varphi)(i) = \inf_{j\in J}\{\mathrm{eq}_I(\alpha(j), i) \to \varphi(j)\}$ with $i \in I$.

Observation 8.12. We observe first that the definition above makes sense: we want to show that if $\varphi \vdash \varphi'$ for $\varphi, \varphi' \in A^J$, then $\forall_A\alpha(\varphi) \vdash \forall_A\alpha(\varphi')$. In other words, if there is an $f \in \Phi$ such that for all $j \in J$, $f\varphi(j) \le \varphi'(j)$, then there exists a $g \in \Phi$ such that $g(\inf_{j\in J}\{\mathrm{eq}_I(\alpha(j), i) \to \varphi(j)\}) \le \inf_{j\in J}\{\mathrm{eq}_I(\alpha(j), i) \to \varphi'(j)\}$ for all $i \in I$. Using the fact that $\to$ is monotone in the second variable we have that: $\inf_{j\in J}\{\mathrm{eq}_I(\alpha(j), i) \to \varphi'(j)\} \ge \inf_{j\in J}\{\mathrm{eq}_I(\alpha(j), i) \to f\varphi(j)\}$. Using Corollary 5.22, (3) we deduce that for some $g \in \Phi$ –in fact in accordance with the mentioned corollary, $g = F(e)F(f) \in \Phi$– we have that $\inf_{j\in J}\{\mathrm{eq}_I(\alpha(j), i) \to f\varphi(j)\} \ge g \inf_{j\in J}\{\mathrm{eq}_I(\alpha(j), i) \to \varphi(j)\}$. Putting both inequalities together we deduce that

$$g \inf_{j\in J}\{\mathrm{eq}_I(\alpha(j), i) \to \varphi(j)\} \le \inf_{j\in J}\{\mathrm{eq}_I(\alpha(j), i) \to \varphi'(j)\},$$

that is our conclusion.

Next we prove that for an arbitrary $\alpha : J \to I$ the map $\forall_A(\alpha) : A^J \to A^I$ is a "right adjoint" of $\alpha^* = \mathcal{R}_A(\alpha) : A^I \to A^J$ with respect to the orden $\vdash$.

Theorem 8.13. *Assume that A is a ${}^{\mathcal{K}}\mathcal{OCA}$. If $I, J \in \mathrm{Set}$, $\alpha : J \to I$ is a function and $\varphi \in A^J$, $\psi \in A^I$, then:*

$$\alpha^*(\psi) \vdash \varphi \Leftrightarrow \psi \vdash \forall_A \alpha(\varphi).$$

Proof. $\Longrightarrow$ From the hypothesis, we deduce that there is an element $f \in \Phi$, with the property that for all $j \in J$ $f\psi(\alpha(j)) \le \varphi(j)$. We take a general $i \in I$, and prove first that for all i, j we have that: $E(f)\psi(i)\,\mathrm{eq}_I(\alpha(j), i) \le \mathrm{i}(f\psi(\alpha(j))) \le f\psi(\alpha(j)) \le \varphi(j)$.

- If $i \neq \alpha(j)$ we deduce from Theorem 5.20,(5.20.20) and Lemma 8.5 that in this situation $E(f)\psi(i)\,\mathrm{eq}_I(\alpha(j), i) \le$ $(f\psi(i)) = \perp \le \varphi(j)$.
- If $i = \alpha(j)$, we deduce similarly that $E(f)\psi(i)\,\mathrm{eq}_I(\alpha(j), i) \le \mathrm{i}(f\psi(\alpha(j))) \le f\psi(\alpha(j)) \le \varphi(j)$.

Hence, using the basic property of the adjunctor, we see that: $\mathrm{e}(E(f)\psi(i)) \le (\mathrm{eq}_I(\alpha(j), i) \to \varphi(j))$. Using as before Theorem 5.20,(5.20.21), we obtain that $(F(\mathrm{e})E(f))\psi(i) \le (\mathrm{eq}_I(\alpha(j), i) \to \varphi(j))$, and taking $\inf_j$ we deduce that if we call $g = F(\mathrm{e})E(f) \in \Phi$, we have that:

$$g\psi(i) \le \forall_A \alpha\varphi(i) \text{ for all } i \in I, \quad \text{i.e.} \quad \psi \vdash \forall_A \alpha(\varphi).$$

$\Longleftarrow$ Our hypothesis guarantees the existence of an element $f \in \Phi$ such that for all i, j we have: $f.\psi(i) \le (\mathrm{eq}_I(\alpha(j), i) \to \varphi(j))$. In particular if $i = \alpha(j)$ we have that for all $j \in J$, $f.\psi(\alpha(j)) \le (\mathrm{i} \to \varphi(j))$ and then, by the basic (half) adjunction condition we see that $(f\psi(\alpha(j)))\,\mathrm{i} \le \varphi(j)$. Using Theorem 5.20,(5.20.23), we obtain that: $M(f, \mathrm{i})\psi(\alpha(j)) \le \varphi(j)$ with $M(f, \mathrm{i}) \in \Phi$ or in other words, we obtain that for all $j \in J$, $M(f, \mathrm{i})\alpha^*(\psi)(j) \le \varphi(j)$ that is what we wanted to conclude.

□

36. We want to prove the so called theorem of Beck–Chevalley.

Theorem 8.14. *Assume that A is a ${}^{\mathcal{K}}\mathcal{OCA}$ and that the following is a pull back diagram in the category of sets:*

$$\begin{array}{ccc} P & \xrightarrow{\rho} & J \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\alpha} \\ K & \xrightarrow[\beta]{} & I \end{array}$$

and consider the corresponding diagram that follows:

$$\begin{array}{ccc} A^P & \xleftarrow{\rho^*} & A^J \\ {\scriptstyle\forall\pi}\downarrow & & \downarrow{\scriptstyle\forall\alpha} \\ A^K & \xleftarrow[\beta^*]{} & A^I \end{array}$$

Then, the second diagram commutes in the sense that for all $\varphi \in A^J$:

$$\beta^*(\forall\alpha\varphi) \vdash \forall\pi(\rho^*(\varphi)) \quad \textit{and} \quad \forall\pi(\rho^*(\varphi)) \vdash \beta^*(\forall\alpha\varphi).$$

Proof. (1) The proof that $\beta^*(\forall\alpha(\varphi)) \vdash \forall\pi(\rho^*(\varphi))$ follows from general categorical properties. We start with the counit relation in Theorem 8.13 that guarantees that $\alpha^*\forall\alpha(\varphi) \vdash \varphi$ and applying ρ^* deduce that $\rho^*\alpha^*\forall\alpha(\varphi) \vdash \rho^*(\varphi)$. From the functoriality of $\mathcal{R}$ we obtain that $\pi^*\beta^*\forall\alpha(\varphi) \vdash \rho^*(\varphi)$, and by Observation 8.12 we get: $\forall\pi\pi^*\beta^*\forall\alpha(\varphi) \vdash \forall\pi\rho^*(\varphi)$. Finally, using the unit of the adjunction in Theorem 8.13 we conclude that $\beta^*\forall\alpha(\varphi) \vdash \forall\pi\pi^*\beta^*\forall\alpha(\varphi) \vdash \forall\pi\rho^*(\varphi)$.

(2) Now we prove that $(\forall\pi(\rho^*(\varphi)) \vdash \beta^*(\forall\alpha\varphi))$. We fix $k_0 \in K$ and need to find an element $g \in \Phi$ such that for all $j \in J$ we have that:

$$g \inf_{z \in P}\{\mathrm{eq}_K(\pi(z), k_0) \to \varphi(\rho(z))\} \le (\,\mathrm{eq}_I(\alpha(j), \beta(k_0)) \to \varphi(j)).$$

We distinguish two possiblities considering if there is an element $z_0 \in P$ such that $\pi(z_0) = k_0$ or not.

- Suppose that we take $z_0 \in P$ with the property that $\pi(z_0) = k_0$, i.e. $z_0 \in \pi^{-1}(k_0)$. In this situation $\mathrm{eq}_K(\pi(z_0), k_0) \to \varphi(\rho(z_0)) = \mathrm{i} \to \varphi(\rho(z_0))$ and it follows from Observation 8.6 and using the notation there, that $g(\,\mathrm{eq}_K(\pi(z_0), k_0) \to \varphi(\rho(z_0))) \leq (\,\mathrm{eq}_K(\pi(z_0), k_0) \to \varphi(\rho(z_0)))$
  Now, given an arbitrary $j \in K$ it may happen that $\rho(z_0) = j$ or $\rho(z_0) \neq j$. In the first case we have that $\alpha(j) = \beta(k_0)$ and that means that $\mathrm{eq}_I(\alpha(j), \beta(k_0)) \to \varphi(j) = \mathrm{i} \to \varphi(\rho(z_0))$. Hence in this case we have that $g(\mathrm{eq}_K(\pi(z_0), k_0) \to \varphi(\rho(z_0))) \leq (\,\mathrm{eq}_I(\alpha(j), \beta(k_0)) \to \varphi(j))$. Otherwise, if $\rho(z_0) \neq j$, we cannot have that $\alpha(j) = \beta(k_0)$ as can be deduced by the basic properties of the pull back. Hence, we have that $\mathrm{eq}_I(\alpha(j), \beta(k_0)) \to \varphi(j) = \perp\to \varphi(j)$ and we obtain again that $g(\mathrm{eq}_K(\pi(z_0), k_0) \to \varphi(\rho(z_0))) \leq (\,\mathrm{eq}_I(\alpha(j), \beta(k_0)) \to \varphi(j))$ from Observation 8.6 where we proved that $g(\mathrm{i} \to \varphi(\rho(z_0))) \leq (\perp\to c)$ for all $c \in A$.
  Hence we have that for all $j \in J$,
  $g \inf_{z\in P}\{\mathrm{eq}_K(\pi(z), k_0) \to \varphi(\rho(z))\} \leq g(\mathrm{eq}_K(\pi(z_0), k_0) \to \varphi(\rho(z_0))) \leq \mathrm{eq}_I(\alpha(j), \beta(k_0)) \to \varphi(j)$.
- Suppose that $\emptyset = \pi^{-1}(k_0) \subseteq P$. In that case is clear that there is no pair $(j, k_0) \in J \times K$ such that $\alpha(j) = \beta(k_0)$ –this follows directly from the fact that the diagram of sets is a pullback. Hence, the inequality to be proved states that for all $j \in J$:
  $$g \inf_{z\in P}\{\perp\to \varphi(\rho(z))\} \leq (\,\perp\to \varphi(j)).$$
  The validity of these type of inequalities is the content of Observation 8.6, (2).

□

37. We want to prove the existence of a generic predicte.

Definition 8.15. Let A be a ${}^{\mathcal{K}}\mathcal{OCA}$[7]. The maps of the form $\mathcal{R}_A(\alpha) : A^I \to A^J$ for $\alpha : J \to I$ are called *reindexing* maps.

A pair (T, Σ) with $T \in A^\Sigma$ is called a *generic predicate* if for all pairs (φ, I) with $I \subset A$ and $\varphi \in A^I$, there is a morphism $\alpha : I \to \Sigma$ such that $\alpha^*(T) = \varphi$.

Theorem 8.16. *In the context of an* inf*–complete $\mathcal{OCA}$, a generic predicate exists.*

Proof. Just take $\Sigma = A$ and $T \in A^A$ the identity map $T = \mathrm{id}_A : A \to A$. It is clear that if $\varphi : I \to A$, then $\varphi^*(T) = \mathrm{id}_A \circ \varphi = \varphi$. □

9. Internal realizability in ${}^{\mathcal{K}}\mathcal{OCA}$s

38. We have shown that the class of ordered combinatory algebras that, besides a filter of distinguished truth values are equipped with an implication, an adjunctor and satisfy a completeness condition with respect to the infimum over arbitrary subsets – i.e.: ${}^{\mathcal{K}}\mathcal{OCA}$s– is rich enough as to allow the Tripos construction and as such its objects can be taken as the basis of the categorical perspective on classical realizability –à la Streicher–. In this section we show that we can define realizability in this type of combinatory algebras, and thus, to define realizability in high order arithmetic.

Definition 9.1. Consider a set of constants of kinds, one of its elements is denoted by o. The language of kinds is given by the following grammar:

$$\sigma, \tau ::= c \quad | \quad \sigma \to \tau$$

Consider an infinite set of variables labelled by kinds x^τ. Suppose that we have infinitely many variables labelled of the kind τ for each kind τ. Consider also a set of constants $a^\tau, b^\sigma, \ldots$ labelled with a kind. The language $\mathcal{L}^\omega$ of order ω is defined by the following grammar:

$$M^\sigma, N^{\sigma\to\tau}, A^o, B^o ::= x^\sigma \quad | \quad a^\sigma \quad | \quad (\lambda x^\sigma.M^\tau)^{\sigma\to\tau} \quad | \quad (N^{\sigma\to\tau}M^\sigma)^\tau \quad | \quad (A^o \Rightarrow B^o)^o \quad | \quad (\forall x^\tau.A^o)^o$$

o represents the type of truth values. The expressions labelled by o are called "formulæ". The symbols $\to$ and $\Rightarrow$, when itereted, are associated on the right side. On the other hand, the application, when iterated, are associated on the left side.

[7]Observe that for this definition and for the theorem that follows, the inf–completeness of A is unnecessary.

Definition 9.2. Consider a ${}^{\mathcal{K}}\mathcal{OCA}$ A and a set of variables $\mathcal{V} = \{x_1, x_2, \dots\}$. A declaration is a string of the shape $x_i : A^o$. A context is a string of the shape $x_1 : A^o_1, \dots, x_k : A^o_k$, i.e.: contexts are finite sequences of declarations. The contexts will be often denoted by capital greek letters: Δ, Γ, Σ. A sequent is a string of the shape $x_1 : A^o_1, \dots, x_k : A^o_k \vdash p : B^o$ where p is a polynomial of $A[x_1, \dots, x_k]$. The left side of a sequent is a context. When we do not explicite the declarations of the context of a sequent, we will write it as $\Gamma \vdash p : B^o$. Typing rules are trees of the shape

$$\frac{S_1 \qquad \dots \qquad S_h}{S_{h+1}}\ \text{(Rule)}$$

where $h \geq 0$ and $S_1, \dots, S_{h+1}$ are sequents. The typing rules for $\mathcal{L}^\omega$ are the following:

$$\text{(where } x_i : A^o_i \text{ appears in } \Gamma\text{)}\ \frac{}{\Gamma \vdash x_i : A^o_i}\ \text{(ax)}$$

$$\frac{\Gamma, x : A^o \vdash p : B^o}{\Gamma \vdash e(\lambda^* x\ p) : (A^o \Rightarrow B^o)^o}\ (\to_i)$$

$$\frac{\Gamma \vdash p : (A^o \Rightarrow B^o)^o \qquad \Gamma \vdash q : A^o}{\Gamma \vdash pq : B^o}\ (\to_e)$$

$$\text{(where } x^\sigma \text{ does not appears free in } \Gamma\text{)}\ \frac{\Gamma \vdash p : A^o}{\Gamma \vdash p : (\forall x^\sigma A^o)^o}\ (\forall_i)$$

$$\frac{\Gamma \vdash p : (\forall x^\sigma A^o)^o}{\Gamma \vdash p : (A^o\{x^\sigma := M^\sigma\})}\ (\forall_e)$$

Definition 9.3. Let us consider $\mathcal{A} = (A, \leq, \circ, \mathrm{s}, \mathrm{k}, \to, \mathrm{e}, \Phi, \inf)$[8] a complete ${}^{\mathcal{K}}\mathcal{OCA}$. We define the interpretation of $\mathcal{L}^\omega$ as follows:

(1) For *kinds*: The interpretation of a constant c is a set $[\![c]\!]$. In particular, the constant o is interpreted as the underlying set of $\mathcal{A}$, i.e.: $[\![o]\!] = A$. Given two kinds σ, τ, the interpretation $[\![\sigma \to \tau]\!]$ is the space of functions $[\![\tau]\!]^{[\![\sigma]\!]}$

(2) For *expressions*: In order to interpret expressions, we start choosing an assignment $\mathfrak{a}$ for the variables x^σ such that $\mathfrak{a}(x^\sigma) \in [\![\sigma]\!]$. As it is usual in semantics, the substitution-like notation $\{x^\sigma := s\}$ affecting an assignment $\mathfrak{a}$ modifies it by redefining $\mathfrak{a}$ over x^σ as the statement $\mathfrak{a}\{x^\sigma := s\}(x^\sigma) := s$. We proceed similarly for interpretations.
   - For an expression of the shape x^σ, its interpretation is $[\![x^\sigma]\!] = \mathfrak{a}(x^\sigma)$.
   - For an expression of the shape $\lambda x^\sigma M^\tau$, its interpretation is the function $[\![\lambda x^\sigma M^\tau]\!] \in [\![\sigma \to \tau]\!]$ defined as $[\![\lambda x^\sigma M^\tau]\!](s) := [\![M^\tau]\!]\{x^\sigma := s\}$ for all $s \in [\![\sigma]\!]$.
   - For an expression of the shape $(N^{\sigma\to\tau} M^\sigma)^\tau$ its interpretation is $[\![(N^{\sigma\to\tau} M^\sigma)^\tau]\!] := [\![N^{\sigma\to\tau}]\!]([\![M^\sigma]\!])$
   - For an expression of the shape $(A^o \Rightarrow B^o)^o$ its interpretation is $[\![(A^o \Rightarrow B^o)^o]\!] := [\![A^o]\!] \to [\![B^o]\!]$.
   - For an expression of the shape $(\forall x^\sigma A^o)^o$ its interpretation is

$$[\![(\forall x^\sigma A^o)^o]\!] := \inf\left\{[\![A^o]\!]\{x^\sigma := s\} \;\middle|\; s \in [\![\sigma]\!]\right\}$$

We say that $\mathcal{A}$ satisfies a sequent $x_1 : A^o_1, \dots, x_k : A^k \vdash p : B^o$ if and only if for all assignment $\mathfrak{a}$ and for all $b_1, \dots, b_k \in A$, if $b_1 \leq [\![A^o_1]\!], \dots, b_k \leq [\![A^o_k]\!]$ then $p\{x_1 := b_1, \dots, x_k := b_k\} \leq [\![B^o]\!]$. In this case we write that: $\mathcal{A} \models x_1{:}A^o_1, \dots, x_k{:}A^k \vdash p{:}B^o$.

A rule:

$$\frac{S_1 \qquad \dots \qquad S_h}{S_{h+1}}\ \text{(Rule)}$$

is said to be *adequate* if and only if for every $\mathcal{A} \in {}^{\mathcal{K}}\mathcal{OCA}$, if $\mathcal{A} \models S_1, \dots, S_h$ then $\mathcal{A} \models S_{h+1}$.

Theorem 9.4. *The rules of the typing system appearing in Definition 9.2, are adequate.*

[8] At this point we must be more precise and distinguish notationally the $\mathcal{OCA}$ $\mathcal{A}$ from its underlying set A.

Proof. For (ax) is evident.

For the implication rules:

$(\to)_i$ Assume $\mathcal{A} \models \Gamma, x : A^o \vdash p : B^o$ where $\Gamma = x_1 : A_1^o, \dots, x_k : A_k^o$. Consider an assignment $\mathfrak{a}$ and $b_1, \dots, b_k \in A$ such that $b_i \leq [\![A_i^o]\!]$. We get:

$$\begin{aligned}(\lambda^* xp)\{x_1 := b_1, \dots, x_k := b_k\}[\![A^o]\!] \; &= \; (\lambda^* xp\{x_1 := b_1, \dots, x_k := b_k\})[\![A^o]\!] \leq \\ &\; p\{x_1 := b_1, \dots, x_k := b_k, x := [\![A^o]\!]\} \leq \\ &\; [\![B^o]\!]\end{aligned}$$

the last inequality by the assumption $\mathcal{A} \models \Gamma, x : A^o \vdash p : B^o$.
Applying the adjunction property we deduce that $\mathrm{e}(\lambda^* xp)\{x_1 := b_1, \dots, x_k := b_k\} \leq [\![(A^o \Rightarrow B^o)^o]\!]$. Since the above is valid for all the assignments, we conclude $\mathcal{A} \models \Gamma \vdash \mathrm{e}(\lambda^* x\, p) : (A^o \Rightarrow B^o)^o$.

$(\to)_e$ Assume $\mathcal{A} \models \Gamma \vdash p : (A^o \Rightarrow B^o)^o$ and $\mathcal{A} \models \Gamma \vdash q : A^o$ where $\Gamma = x_1 : A_1^o, \dots, x_k : A_k^o$. Consider an assignment $\mathfrak{a}$ and $b_1, \dots, b_k \in A$ such that $b_i \leq [\![A_i^o]\!]$. By hypothesis we get:

$$p\{x_1 := b_1, \dots, x_k := b_k\} \quad \leq \quad [\![A^o]\!] \to [\![B^o]\!]$$

and

$$q\{x_1 := b_1, \dots, x_k := b_k\} \quad \leq \quad [\![A^o]\!]$$

and by monotonicity of the application in $\mathcal{A}$ we obtain:

$$pq\{x_1 := b_1, \dots, x_k := b_k\} \quad \leq \quad ([\![A^o]\!] \to [\![B^o]\!]) \quad [\![A^o]\!] \quad \leq \quad [\![B^o]\!]$$

Since the above is valid for all the assignments, we conclude that $\mathcal{A} \models \Gamma \vdash pq : [\![B^o]\!]$.

For the quantifiers:

$(\forall)_i$ Assume $\mathcal{A} \models \Gamma \vdash p : A^o$ and that x^σ does not appear free in Γ, where $\Gamma = x_1 : A_1^o, \dots, x_k : A_k^o$. Consider an assignment $\mathfrak{a}$ and $b_1, \dots, b_k \in A$ such that $b_i \leq [\![A_i^o]\!]$.
Since $A_1^o, \dots, A_k^o$ does not depend upon x^σ, by the assumption $\mathcal{A} \models \Gamma \vdash p : A^o$, we get:

$$p\{x_1 := b_1, \dots, x_k := b_k\} \leq [\![A^o]\!]\{x^\sigma := s\} \text{ for all } s \in [\![\sigma]\!]$$

Then $p\{x_1 := b_1, \dots, x_k := b_k\} \leq \inf\{[\![A^o]\!]\{x^\sigma := s\} \mid s \in [\![\sigma]\!]\} = [\![(\forall x^\sigma A^o)^o]\!]$. We conclude as before that $\mathcal{A} \models \Gamma \vdash p : (\forall x^\sigma A^o)^o$.

$(\forall)_e$ Assume $\mathcal{A} \models \Gamma \vdash p : (\forall x^\sigma A^o)^o$, where $\Gamma = x_1 : A_1^o, \dots, x_k : A_k^o$. Consider an assignment $\mathfrak{a}$ and $b_1, \dots, b_k \in A$ such that $b_i \leq [\![A_i^o]\!]$. By the assumption $\mathcal{A} \models \Gamma \vdash p : (\forall x^\sigma A^o)^o$ we get:

$$p\{x_1 := b_1, \dots, x_k := b_k\} \leq [\![A^o]\!]\{x^\sigma := s\} \text{ for all } s \in [\![\sigma]\!]$$

Since $[\![M^\sigma]\!] \in [\![\sigma]\!]$ we obtain:

$$p\{x_1 := b_1, \dots, x_k := b_k\} \leq [\![A^o]\!]\{x^\sigma := [\![M^\sigma]\!]\} = [\![A^o\{x^\sigma := M^\sigma\}]\!]$$

We conclude as before that $\mathcal{A} \models \Gamma \vdash p : A^o\{x^\sigma := M^\sigma\}$.

□

The language of high order Peano Arithmetics –(PA)$^\omega$–is an instance of $\mathcal{L}^\omega$ where we distinguish a constant of kind I and two constants of expression 0^I and $\mathrm{succ}^{I \to I}$.

Definition 9.5. For each kind σ we define the Leibniz equality $=_\sigma$ as follows:

$$x_1^\sigma =_\sigma x_2^\sigma :\equiv \forall y^{\sigma \to o}\Big((y^{\sigma \to o} x_1^\sigma)^o \Rightarrow (y^{\sigma \to o} x_2^\sigma)^o\Big)^o$$

The axioms of Peano Arithmetics are equalities over the kind I, except for $\forall x^I((\mathrm{succ}^{I \to I}\, x^I =_I 0^I) \Rightarrow \bot)^o$ –which we abbreviate $\forall x^I(\mathrm{succ}^{I \to I}\, x^I \neq 0^I)^o$– and for the induction principle.

From the work of Krivine (c.f.: [6]) we can conclude that all Peano Axioms except the induction principle are realized in every ${}^{\mathcal{K}}\mathcal{OCA}$.

Lemma 9.6. *All equational axioms of Peano Arithmetics are realized in every* ${}^{\mathcal{K}}\mathcal{OCA}$.

Proof. For the axioms which are equalities, the identity term $\lambda^* xx$ suffices as a realizer. For the axiom $\forall x^I(\mathrm{succ}^{I \to I}\, x^I \neq 0^I)$, the term $\lambda^* x\, x\, \mathrm{s}$ is a realizer. □

Definition 9.7. The formula $\mathbb{N}(z^I)$ is defined as:

$$\forall x^{I\to o}(\forall y^I((x^{I\to o}y^I)^o \Rightarrow (x^{I\to o}(\text{succ}^{I\to I}\, y^I))^o \Rightarrow ((x^{I\to o}0^I)^o \Rightarrow (x^{I\to o}z^I)^o)^o$$

The meaning of this definition is that $\mathbb{N}(z^I)$ is satisfied in the sort I by the individuals z^I which are in all the inductive sets.

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Facultad de Ciencias, Universidad de la República, Iguá 4225, 11400. Montevideo, Uruguay,
E-mail address: wrferrer@cmat.edu.uy

Facultad de Ingeniería, Universidad de la República, J. Herrera y Reissig 565, 11300. Montevideo, Uruguay,
E-mail address: mguille@fing.edu.uy

Facultad de Ingeniería, Universidad de la República, J. Herrera y Reissig 565, 11300. Montevideo, Uruguay,
E-mail address: malherbe@fing.edu.uy